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On n-tardy sets

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ABSTRACT

Harrington and Soare introduced the notion of an n-tardy set. They showed that there is a nonempty $\mathcal{E}$ property $Q(A)$ such that if $Q(A)$ then A is 2-tardy. Since they also showed no 2-tardy set is complete, Harrington and Soare showed that there exists an orbit of computably enumerable sets such that every set in that orbit is incomplete. Our study of n-tardy sets takes off from where Harrington and Soare left off. We answer all the open questions asked by Harrington and Soare about n-tardy sets. We show there is a 3-tardy set A that is not computed by any 2-tardy set B. We also show that there are nonempty $\mathcal{E}$ properties $Q_n(A)$ such that if $Q_n(A)$ then A is properly n-tardy.

1. Introduction

Let $\mathcal{E}$ denote the structure of c.e. sets under the language of inclusion. Understanding the interplay between computability and definability in $\mathcal{E}$ is a longstanding area of research in classical computability theory. In 1944 [8], Post set out to find an incomplete noncomputable c.e. set, i.e., a noncomputable c.e. set that does not have the degree of the halting problem K. He defined several properties of c.e. sets (such as *simplicity*) in the hope that no c.e. set satisfying one of these properties could be complete. All of the properties he suggested failed to satisfy this condition, but many of them are definable in $\mathcal{E}$. Although Friedberg and Mučnik [7,2] famously obtained an incomplete noncomputable c.e. set using a priority argument, a natural question is whether there exists an $\mathcal{E}$-definable nontrivial property Q such that if $Q(A)$ holds, then A is an incomplete noncomputable c.e. set. Harrington and Soare produced such a property Q in [4], and they also described an $\mathcal{E}$-definable property that guarantees completeness (see [9], p. 339 and [3]). These results are part of work by many towards the following general goal.

Question 1.1. *Characterize what sets are and are not automorphic to a complete set.*

Harrington and Soare showed that all sets that satisfy Q are 2-*tardy* [6], a slowness condition that we describe, along with the conditions n-*tardy* and *very tardy*, in Section 1.2. The very tardy sets, by definition, are those that are not almost prompt, and all complete sets are prompt. All n-tardy sets are very tardy and, hence, incomplete. Thus, any A for which $Q(A)$ holds is not automorphic to a complete set. On the other hand, Harrington and Soare [5], building on work of the first author,

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Downey, and Stob [1], proved that every almost prompt set (i.e., every not very tardy set) is automorphic to a complete set. Thus, in order to work towards answering Question 1.1, we explore the very tardy sets and their orbits from the perspectives of computability and definability. We begin by defining varying notions of promptness.

1.1. Prompt and almost prompt sets

Definition 1.2. 1. A coinfinite c.e. set A is *promptly simple* if there is a computable function p and a computable enumeration $\{A_s\}_{s\in\omega}$ of A such that for every $e \in \omega$,

$$W_e \text{ infinite } \implies (\exists s)(\exists x)[x \in W_{e,\,\text{at}\,s} \cap A_{p(s)}].$$

2. A c.e. set A is *prompt* if A has promptly simple degree, i.e., $A \equiv_T B$ for some promptly simple set B, and a c.e. degree is *prompt* if it contains a prompt set.

Definition 1.3. A set X is n-c.e. iff there is a computable sequence of c.e. sets $\{W_{e_i} : 1 \le i \le n\}$ such that

$X = (W_{e_1} - W_{e_2}) \cup \cdots \cup (W_{e_{n-2}} - W_{e_{n-1}}) \cup W_{e_n}$ if n is odd, and
$X = (W_{e_1} - W_{e_2}) \cup \cdots \cup (W_{e_{n-1}} - W_{e_n})$ if n is even.

The sequence of sets $\{W_{e_i} : 1 \le i \le n\}$ is an *n-c.e. presentation of* X. Such a sequence can be used to give a stagewise approximation of X:

$$X_s = (W_{e_1,s} - W_{e_2,s}) \cup \cdots .$$

For $n > 0$, we define the computable enumeration $\{X_e\}_{e\in\omega}$ of all n-c.e. sets so that if $e = \langle e_1, e_2, \ldots, e_n\rangle$,

$$X_e^n = (W_{e_1} - W_{e_2}) \cup \cdots$$

and $X_{e,s}^n$ denotes the stagewise approximation of X_e^n.

Definition 1.4 (*Definition 11.3 of [5]*)**.** Let A be a c.e. set and $\{A_s\}$ be an enumeration of A. The set A is *almost prompt* iff there is a nondecreasing function $p(s)$ such that for all n and all e

$$X_e^n = \overline{A} \implies (\exists x)(\exists s)[x \in X_{e,s}^n \wedge x \in A_{p(s)}]. \tag{1.5}$$

Harrington and Soare [5] showed that this definition is robust. That is, if Eq. (1.5) holds for some enumeration of A, it holds for all enumerations of A (see [5, Theorem 11.4]). They also proved that any c.e. set of prompt degree is almost prompt (see [5, Theorem 11.7]); thus, the notion of almost prompt generalizes the notion of prompt. They also showed that almost prompt sets are ubiquitous in the following sense.

Theorem 1.6 (*Harrington and Soare, Theorem 11.12 [5]*)**.** *There are almost prompt sets of every c.e. degree.*

Moreover, they showed that there are tardy (i.e., not of prompt degree) sets A such that every degree Turing above A is almost prompt, (see [5, Theorem 11.8]) and that the join of an almost prompt set and any computably enumerable set is almost prompt (see [5, Theorem 11.11]).

In order to show Theorem 1.6, Harrington and Soare proved that every low simple set is almost prompt (see [5, Theorem 11.10]). Recall that a set is *low*$_2$ if $A'' \le_T \emptyset''$ where X' denotes the halting jump of the set X. Harrington and Soare left the following question open:

Question 1.7 (*Question 1 of [5]*)**.** *If A is low$_2$ and simple, is A almost prompt?*

We provide a negative answer to Question 1.7 in Section 5, but we first focus on particular classes of sets that are not almost prompt. These sets are of particular importance to Question 1.1 because of the following theorem.

Theorem 1.8 (*Harrington and Soare [5], extending Cholak et al. [1]*)**.** *Every almost prompt set is automorphic to a complete set.*

1.2. Very tardy and n-tardy sets

A degree is *tardy* if it is not a prompt degree. A set is *very tardy* if it is not almost prompt. (Note that being very tardy is a property of sets and does not readily extend to degrees.) Since the definition of almost prompt is robust, we have the following equivalent definition.

Definition 1.9. Let A be c.e. and $\{A_s\}$ be an enumeration of A. The set A is *very tardy* iff A is not almost prompt iff for every nondecreasing computable function $p(s)$ there is an n and an e such that

$$X_e^n = \overline{A} \,\&\, (\forall x)(\forall s)[x \in X_{e,s}^n \implies x \notin A_{p(s)}]. \tag{1.10}$$

Moreover, A is *n-tardy* iff there is a single n that works for all such functions $p(s)$, and A is *properly n-tardy* if A is n-tardy but not $n-1$-tardy.

By Theorem 1.8, any potential example of a set that is not automorphic to a complete set must be very tardy. As described in the introduction, Harrington and Soare found such a set.

Theorem 1.11 (*Harrington and Soare [4]*). *There exists an $\mathcal{E}$-definable nontrivial property Q such that if $Q(A)$ holds, then A is not automorphic to a complete set.*

We will state the property Q in Definition 3.1, however it is easiest to understand the property Q in terms of 2-tardy sets. Specifically, Harrington and Soare showed that Q describes a particular subset of the 2-tardy sets. We need a few definitions in order to define this subset.

Definition 1.12 (*See [9] Definitions 4.1 and 4.10*). 1. Let $A \subset_\infty C$ denote that $A \subset C$ and $C - A$ is infinite.
2. A subset A is a *major* subset of C if $A \subset_\infty C$ and for all e,

$$\overline{C} \subseteq W_e \implies \overline{A} \subseteq^* W_e.$$

3. A subset $A \subset C$ is a *small* subset of C (written $A \subset_s C$)) if $A \subset_\infty C$ and for all X and Y,

$$X \cap (C - A) \subseteq Y \implies (\exists Z)_{Z \subseteq X}[Z \supseteq (X - C)\ \&\ (Z \cap C) \subseteq Y].$$

4. If A is both a small subset and a major subset of C, we call A a *small major* subset of C and write $A \subset_{sm} C$.

Theorem 1.13 (*Harrington and Soare [6] Corollary 3.10*). $Q(A) \iff (\exists C)[A \subset_{sm} C\ \&\ A$ *is* 2-*tardy*].

Harrington and Soare used this characterization to show that any A satisfying $Q(A)$ is not automorphic to a complete set.

Definition 1.14. The *orbit* of A, denoted by $[A]$, is the set of c.e. sets B such that there exists an automorphism Ψ of $\mathcal{E}$ sending A to B.

If A satisfies $Q(A)$ and there is an automorphism Ψ of $\mathcal{E}$, then $Q(\Psi(A))$ holds as well. In other words, Q holds of any element in $[A]$. Since Q holds of all sets in $[A]$, all sets in $[A]$ are 2-tardy and therefore incomplete. Thus, if A satisfies $Q(A)$, A is not automorphic to a complete set.

In Section 3, we define nontrivial properties $\hat{Q}_n$ that generalize Q. In Theorem 3.2, we show that if $\hat{Q}_n(A)$ holds, then A is n-tardy and $\neg\hat{Q}_i(A)$ holds for all $i < n$. In Theorem 4.1, we show that there is some properly n-tardy set A_n for which $\hat{Q}_n(A_n)$ holds. Thus, the collection $\{[A_n]\}_{n\in\omega}$ witnesses that the c.e. sets that are not automorphic to a complete set break into countably many disjoint orbits.

1.3. Codable sets

In [6], Harrington and Soare also explored the connection between tardiness and what sets X are coded in every nontrivial orbit in the following sense.

Definition 1.15 (*[6] Definition 1.3*). 1. We say X is *coded in the orbit of* A, denoted $X \leq_T [A]$, if $X \leq_T B$ for some $B \in [A]$.
2. We say X is *codable* if for every noncomputable set A, $X \leq_T [A]$.

Harrington and Soare obtained the following characterization of the codable sets by using the Δ^0_3-automorphism method they developed in [3].

Theorem 1.16 (*Harrington and Soare [6] Corollary 1.8*). *A set is codable iff $X \leq_T D$ for some D satisfying $Q(D)$.*

Using Theorem 1.16, Harrington and Soare obtain the following simple corollary.

Corollary 1.17 (*Harrington and Soare [6] Corollary 1.9*). *If S has prompt degree, then S is not codable.*

Harrington and Soare in fact showed that a set is codable iff $X \leq_T D$ for some 2-tardy D; Theorem 1.16 only uses the fact that if $Q(D)$ holds, then D is 2-tardy. Thus, the ability to code in the above sense is more connected to enumeration speed than degree-theoretic content. Given this observation, it is natural to wonder whether all very tardy sets are codable. Harrington and Soare asked a more specific version of this problem:

Question 1.18 (*Harrington and Soare [5] Question 1*). *Are all* 3-*tardy sets codable?*

By Theorem 1.16, this is equivalent to the following question.

Question 1.19. *If A is* 3-*tardy, does there exist a* 2-*tardy set B such that $A \leq_T B$?*

Let A be 2-tardy. If $A_0 \sqcup A_1 = A$ is a nontrivial split, then each of the A_i are 3-tardy. To see this, given a nondecreasing computable function $p(s)$, there is an X^2_e witnessing that Eq. (1.10) holds for A. Then, $X^3_{\tilde{e}} = (X^2_{e_1} - X^2_{e_2}) \sqcup A_{\bar{i}}$ witnesses that Eq. (1.10) holds for A_i with respect to $p(s)$, where $\bar{0} = 1$ and $\bar{1} = 0$. Prior to this work, it was unknown whether every 3-tardy is the split of a 2-tardy. If this was the case, then clearly every 3-tardy would be computable from the 2-tardy of which it is a split, and hence would be codable. In Section 2, we show that not all 3-tardy sets are splits of 2-tardy sets. In fact, we answer Question 1.19 negatively. We show that there exists a 3-tardy set that is not computed by any 2-tardy. Hence, not all 3-tardy (and very tardy) sets are codable.

2. A 3-tardy not computed by any 2-tardy

We devote this section to constructing a 3-tardy set A not computed by any 2-tardy set. Hence, A is noncodable.

Theorem 2.1. *There is a* 3*-tardy set A such that for all* 2*-tardy sets B,* $A \not\leq_T B$*.*

Proof. We will construct A. Our construction style will be a pinball machine laid out on top of a tree. Here our tree will be $3^{<\omega}$, and given two nodes α, β we say $\alpha <_L \beta$ if $\alpha\upharpoonright_l$ occurs lexicographically before $\beta\upharpoonright_l$ for some l. Since balls move downward (gravity) in this case we want to think of our tree as growing upward. As always, we are most concerned about the action of the pinball machine along the true path f. We will have an approximation f_s of the true path f, such that $f = \liminf_s f_s$. We say that a node α is *visited* at stage s if $\alpha \preceq f_s$ and is *reset* at stage $s+1$ if $f_s <_L \alpha$.

The approximation to the true path, f_s, will help determine the movement of the balls (integers) on the pinball machine. Balls will be placed on the pinball machine by a node $\alpha \prec f_s$ at stage s only when we wish to put them into A. At stage s, all balls x on the machine will be located at some node $\alpha(x, s)$. If $\alpha(x, s) = \lambda$ (λ is the empty node), we put x into A_{s+1} and remove x from the machine at stage $s+1$. So when a ball x is on the machine our apparent goal is to move x downward and into A. At some point later, we will *sometimes* change our mind and remove balls from the machine, preventing them from going into A. If $f_{s+1} <_L \alpha(x, s)$ we will also remove x from the machine at stage $s+1$ and never use it again. At stage $s+1$, we are free to place any ball $x \leq s+1$ that has never been used on the machine. However, we must ensure that for all s, if $\alpha(x, s)\downarrow$ then, for all t such that $x \leq t \leq s$, $\alpha(x, s) \leq_L f_t$. The action to ensure this goes on at every stage in the background.

Our next goal is to make A a 3-tardy set. This means that balls must enter A very slowly. We have to meet the following requirements:

$$\text{If } \varphi_e \text{ total \& nondecreasing, then } \left(\exists X_e^3\right)\left(X_e^3 = \overline{A} \wedge (\forall x)(\forall s)\left[x \in X_{e,s}^3 \implies x \notin A_{\varphi_e(s)}\right]\right). \qquad (\mathcal{N}_e{:})$$

In general, the way to meet $\mathcal{N}_e$ is to ensure that for all balls x there is a stage s_1 at which we put x into A_{s_1} or $X^3_{e_1,s_1}$. Now if a ball x in $X^3_{e_1,s_1}$ wants to enter A at stage $s_2 > s_1$ we must put x into $X^3_{e_2,s_2}$. Then we wait until a stage s_3 such that $\varphi_{e,s_3}(s_2)\downarrow$. If such a stage s_3 exists then we *must* eventually put x into A or $X^3_{e_3}$. If a ball x is in $X^3_{e_2,s_2}$ and we remove it from the machine at stage s_4, we will put x into $X^3_{e_3}$ at stage s_4. If $\varphi_e(s_2)\uparrow$ then φ_e is not total and the requirement is satisfied.

In the tree construction, we will use node γ to meet $\mathcal{N}_e$. We will label the 3-c.e. set constructed at γ, as X^3_γ rather than X^3_e. At stages s, where $\gamma \preceq f_s$ we will put all balls $x \notin A_s$ such that $|\gamma| \leq x \leq s$ into $X^3_{\gamma_1,s}$. If $\gamma \prec f$ then almost all balls not in A are in $X^3_{\gamma_1}$. At the first stage where $\alpha(x, s) = \gamma$, we will put x into $X^3_{\gamma_2}$. If we remove x from the machine before entering A, we will put x into $X^3_{\gamma_3}$. Should f_s ever be to the left of γ, then some ball x with $\alpha(x, s) \preceq f_s$ already in $X^3_{\gamma_1,s}$ might enter A without proper delay. However, since only finitely many such stages may occur along the true path whenever f_s moves to the left of γ, we may reset our construction of X^3_γ (equivalently, we imagine that the tree guesses at how many elements each positive requirement places into A).

Given a stage $s+1$ such that $\gamma \preceq f_{s+1}$, let $t \leq s$ be the greatest stage such that $\gamma \preceq f_t$ (if t does not exist let $t = 0$). Define

$$l_\gamma(s) = \max x[(\forall z < x)\varphi_{e,s}(z)\downarrow].$$

The function $l_\gamma(s)$ measures the length of convergence of φ_e at stage s. If $l_\gamma(s) > l_\gamma(t)$ and, for all $x \in X^3_{\gamma_2,s}$, if $x \in X^3_{\gamma_2,\text{at } s'}$ then $l_\gamma(s) > s'$, then we say that $s+1$ is γ*-expansionary*. In other words, stage $s+1$ is γ-expansionary if the length of convergence of φ_e has increased and the proper amount of delay for all $x \in X^3_{\gamma_2,s}$ has been determined. At γ-expansionary stages $s+1$, we move all balls x such that $\alpha(x, s) = \gamma$ downward so that $\alpha(x, s+1) = \beta$, where $\beta\hat{}0 \preceq \gamma$ and β is the greatest such subnode of γ assigned to some $\mathcal{N}_{e'}$ (only nodes working on the requirements $\mathcal{N}_{e'}$ stop balls from moving downwards) or if no such β exists let $\beta = \lambda$. If $s+1$ is γ-expansionary, we let $\gamma\hat{}0 \prec f_{s+1}$. Otherwise, we let $\gamma\hat{}1 \prec f_{s+1}$. If we have moved any balls downwards or $|\gamma| = s$, we end this stage. Otherwise, we consider the action of $\gamma\hat{}0$ or $\gamma\hat{}1$.

$\mathcal{N}_e$ is a Π^0_2 requirement. How $\mathcal{N}_e$ is met depends on the answer to the Π^0_2 question is φ_e total. Suppose that $\gamma \prec f$. Define f such that $\gamma\hat{}0 \prec f$ if φ_e is total and $\gamma\hat{}1 \prec f$ if not. If $\gamma \prec f$ then it not hard to see that $\liminf f_s \upharpoonright (|\gamma| + 1) = \gamma\hat{}0$ iff φ_e is total.

Note that we have made the simplifying assumption that if we enumerate x into $X^3_{\gamma_2}$ at stage s then $x \in X^3_{\gamma_2,s}$. While we may simply choose an enumeration of $X^3_{\gamma_2}$ to make this true, we must satisfy $\mathcal{N}_e$ with respect to the canonical enumeration of elements into c.e. sets. However, using the recursion theorem, we may safely assume that each node is actually in possession of an index for every c.e. set built at that node and then, when necessary, we can simply wait until every element enumerated into some $X^3_{\gamma_2}$ appears in it in the canonical enumeration. Since such modifications are straightforward but tedious, we will refrain from further mention of them.

We assume the nodes that place balls on the machine obey the following rules and assumptions. A node $\alpha \succ \gamma$ can only place a ball x on the machine at stage t if $x \in X^3_{\gamma_1,t}$. Moreover, while α might place a ball on the machine at stage s, α can only place these balls at nodes working on the requirement $\mathcal{N}_{e'}$ for some e'. While we will not restrict how many balls α can place on the machine, we assume

Only finitely many balls that α places on the machine enter A. $(\mathcal{A})$

Assume that $\gamma\hat{\ }0 \prec f$. Let s' be such that for all $s \geq s'$, $\gamma \preceq f_{s'}, f_s \not<_L \gamma$, and no $\alpha \prec \gamma$ places any more balls on the machine at stage s that later enter A. Under our extra Assumption $(\mathcal{A})$, we know such a stage exists. Assume we are dealing with stages $s \geq s'$. It is not difficult to verify by induction on the length of γ that if $\gamma\hat{\ }0 \preceq f_s$, $\alpha(x, s-1) = \gamma$, and $t > s$ is the next stage such that $\gamma\hat{\ }0 \preceq f_t$ then either $x \in A_t$ or $x \in X^3_{\gamma_3,t}$ and for all $y > s'$, if $\alpha(y, t) \prec \gamma$ then $\alpha(y, t)\hat{\ }1 \prec \gamma$. It is easy to determine which balls enter A between such stages. We assumed that all balls placed on the machine by nodes $\alpha \prec \gamma$ that enter A have already entered by stage s'. Therefore, the balls that enter A between s' and t come from nodes to the right of γ. Since these nodes were reset at stage s', these balls all have to be larger than s' (otherwise we have that $\alpha(y, t') \leq_L \gamma$ for some stage $s' < t' < t$) and get into A by stage t. Hence, with the above movement of balls and Assumption $(\mathcal{A})$ we have that $X^3_\gamma =^* \overline{A}$ and we have met $\mathcal{N}_e$.

Our next goal is to make A so that it is not computed by any 2-tardy. We must meet the requirements:

If $\Phi_{e_1}(W_{e_2}) = A$, then W_{e_2} is not 2-tardy. $(\mathcal{P}_e{:})$

We will assign a parent node α to $\mathcal{P}_e$. Node α will be working on the requirement:

If $\Phi_\alpha(W_\alpha) = A$ then W_α is not 2-tardy. $(\mathcal{P}_\alpha{:})$

Determining whether $\Phi_\alpha(W_\alpha) = A$ is Π^0_2. So α will have two outcomes 1 and 2: outcome 1 if $\Phi_\alpha(W_\alpha) = A$ and outcome 2 otherwise. We will later use outcome 0 to denote a Σ^0_1 win. Like above, determining whether $\Phi_\alpha(W_\alpha) = A$ can be measured by asking if there are infinitely many expansionary stages where length here measures length of agreement between $\Phi_\alpha(W_\alpha)$ and A.

Assume $\alpha \preceq f_s$. Let $t \leq s$ be the greatest stage such that $\alpha \preceq f_t$ (if t does not exist let $t = 0$). Define $l_\alpha(s) = \max x[(\forall z < x)\Phi^{W_{\alpha,s}}_{\alpha,s}(z)\downarrow = A_s(z)]$.
We say that $s + 1$ is *γ-expansionary* if

1. $l_\alpha(s) > l_\alpha(t)$ and,
2. for all $\beta \succeq \alpha$, if x_β is defined (these will be witnesses to help meet requirement $\mathcal{P}_\beta$) then $l_\alpha(s) > x_\beta$.

If $s + 1$ is α-expansionary, let $\alpha\hat{\ }1 \preceq f_{s+1}$. Otherwise, $\alpha\hat{\ }2 \preceq f_{s+1}$. If there are only finitely many expansionary stages, we need not take any action to meet $\mathcal{P}_\alpha$. We only need to take action if it appears there are infinitely many expansionary stages (the Π^0_2 outcome).

We can define the function $p_\alpha(t) = s$ iff $s > t$ is the least stage such that $\alpha\hat{\ }1 \prec f_s$. If $\alpha\hat{\ }1 \prec f$ then p_α is computable. From our work above, we know if $\alpha(x, t)\hat{\ }0 \prec \alpha$ then at stage $s = p_\alpha(t)$ either x is in A or removed from the machine. This is the function we will try to use to witness W_α is not 2-tardy.

As a first approximation to showing W_α is not 2-tardy, we might try the following. Above the node $\alpha\hat{\ }1$, we will have nodes β working on the requirements:

If $X^2_e = \overline{W}_\alpha$ then there exists y and s such that $y \in X^2_{e,s}$ and $y \in W_{\alpha,p_\alpha(s)}$. $(\mathcal{P}_{\alpha,e}{:})$

The idea to meet $\mathcal{P}_{\alpha,e}$ is the following: At a stage s where $\beta \prec f_s$, choose some large ball x_β. Keep x_β out of A and off the machine. Let $u_{\alpha,s}(x)$ be the use of $\Phi^{W_\alpha}_{\alpha,s}(x)$. Wait for a stage s where $\alpha\hat{\ }1 \preceq f_s$, $X^2_{e,s} \restriction u_{\alpha,s}(x_\beta) = \overline{W}_{\alpha,s} \restriction u_{\alpha,s}(x_\beta)$ and $\beta \prec f_s$. If such a stage can be found, we want to add x_β to A quickly, before stage $t = p_\alpha(s)$. If we can do that, some ball $y < u_{\alpha,s}(x_\beta)$ must enter W_α by stage t since $\Phi^{W_\alpha}_\alpha = A$. That y must be in $X^2_{e,s}$.

The problem is adding these balls into A quickly. If we could place the balls x_β that we want to enter A into the machine at some node $\gamma \preceq \alpha$ at stage s, then by our work above we would not have a problem. Since there might be infinitely many $\mathcal{P}_{\alpha,e}$ that want to place balls into A, this would violate our extra Assumption $(\mathcal{A})$. We might try to remove this extra assumption. But even so, the set of balls that all requirements $\mathcal{P}_{\alpha,e}$ might want to add to A is not computable. So, we have no reasonable way to manage these balls if we allow them all to enter the machine at α or below.

Hence, for each e we must assign a different node $\beta \succeq \alpha\hat{\ }1$ to $\mathcal{P}_{\alpha,e}$. When β wants to add x_β to A, the node β places x_β at the largest substring $\gamma = \nu\hat{\ }0$ of β where ν is assigned to some $\mathcal{N}_{e'}$. Let stage t' be the first stage that x_β goes below α in the machine. At such a stage we have that $\alpha\hat{\ }1 \prec f_{t'}$. If $X^2_{e,t'} \restriction u_{\alpha,t'}(x_\beta) = \overline{W}_{\alpha,t'} \restriction u_{\alpha,t'}(x_\beta)$, we let x_β continue downwards into A for a win (on the above y and t') on $\mathcal{P}_{\alpha,e}$ as described above. But this may no longer be the case. We have no reason to believe that t' is expansionary for $X^2_e = \overline{W}_\alpha$. It may be the case that at stage t', $X^2_{e,t'}$ is already correctly predicting which balls y will enter W_α.

Hence, we must modify our requirements to

If $X^2_e = \overline{W}_\alpha$ and $\neg[(\exists y)(\exists s)[y \in X^2_{e,s} \wedge y \in W_{\alpha,p_\alpha(s)}]]$ $(\mathcal{P}_{\alpha,e}{:})$

then for all i

If $X^2_i = X^2_e$ then $(\exists y)(\exists s)[y \in X^2_{i,s} \wedge y \in W_{\alpha,p_\beta(s)}]]$. $(\mathcal{P}_{\alpha,e,i}{:})$

As before some $\beta \succ \alpha\hat{\ }1$ will be assigned to $\mathcal{P}_{\alpha,e}$. The node β will have three possible outcomes. The first, $\beta\hat{\ }0$, is in the case we have a Σ^0_1 win for $\mathcal{P}_{\alpha,e}$, i.e., a ball y and stage s where $y \in X^2_{e,s}$ and $y \in W_{\alpha,p_\alpha(s)}$. The second outcome $\beta\hat{\ }1$ holds if there is not a Σ^0_1 win and $X^2_e = \overline{W}_\alpha$. The $\beta\hat{\ }2$ outcome holds otherwise. As above, we will measure whether $X^2_e = \overline{W}_\alpha$ by expansionary stages.

Assume $\beta \preceq f_{s+1}$. Let $t \leq s$ be the greatest stage such that $\beta\hat{\ }1 \preceq f_t$ (if t does not exist, let $t = 0$). Define $l_\beta(s) = \max x[(\forall z < x)[X^2_{e,s}(z) = \overline{W}_{\alpha,s}(z)]$. We say that $s+1$ is β-*expansionary* if

1. $l_\beta(s) > l_\beta(t)$ and
2. for all $\delta \succeq \beta$, if x_δ (a ball to satisfy $\mathcal{P}_{\alpha,e,i}$) is defined then $l_\beta(s) > u_{e,s}(x_\delta)$.

If stage $s+1$ is γ-expansionary and we have not seen a Σ^0_1 win for $\mathcal{P}_{\alpha,e}$, then $\beta\hat{\ }1 \preceq f_{s+1}$. If we have seen the Σ^0_1 win, then $\beta\hat{\ }0 \preceq f_{s+1}$. Otherwise, $\beta\hat{\ }2 \preceq f_{s+1}$.

If there are only finitely many expansionary stages or we see the Σ^0_1 win, $\mathcal{P}_{\alpha,e}$ is automatically satisfied. Assume that this is not the case. Hence, as above, we are in Π^0_2 outcome. In this case, we must meet $\mathcal{P}_{\alpha,e,i}$, for all i. For each i we will assign some node $\delta \succeq \beta\hat{\ }1$ to $\mathcal{P}_{\alpha,e,i}$. The outcomes and approximations to the true path for δ are defined in similar fashion to what was done for β and we will not repeat them. The issue for δ is showing that δ does not have the Π^0_2 outcome, $\delta\hat{\ }1$.

At a stage s where $\delta \prec f_s$, choose a large unused ball x_δ, which we hold out of A and the machine. Wait for a stage s where $\delta\hat{\ }1 \preceq f_s$. If such a stage does not exist we have won this requirement. If such a stage exists, then place x_δ into the machine at the largest substring $\gamma = \nu\hat{\ }0$ of β (note, not δ) where ν is assigned to some $\mathcal{N}_{e'}$ and end this stage.

Now, assuming $\beta\hat{\ }1 \prec f$, there will be a later stage t' where x_δ moves below α and $\alpha\hat{\ }1 \preceq f_{t'}$. Otherwise, $\alpha\hat{\ }2 \prec f$ and then the action of α, β and γ are finitary and therefore Assumption $(\mathcal{A})$ holds. If $X^2_{e,t'} \restriction u_{e,t'}(x_\delta) = \overline{W}_{\alpha,t'} \restriction u_{e,t'}(x_\delta)$, we let x_δ continue downwards into A for a win on $\mathcal{P}_{\alpha,e}$ as before. If this happens for any i, it will provide us with a Σ^0_1 win on $\mathcal{P}_{\alpha,e}$ and all the balls x_δ will be removed from the tree since they are to the right of the true path. Therefore, the action of β and those $\delta \succ \beta\hat{\ }1$ will be finitary. Hence, in this case, Assumption $(\mathcal{A})$ holds.

Assume that $X^2_{e,t'} \restriction u_{e,t'}(x_\delta) \neq \overline{W}_{\alpha,t'} \restriction u_{e,t'}(x_\delta)$. Here, we will remove x_δ from the machine. We put x_δ into $X^3_{\gamma,3}$ at stage t', for all γ such that x_δ is in $X^3_{\gamma,2}$ at stage t'. We now have to argue that this provides us with a win for $\mathcal{P}_{\alpha,e,i}$.

Remark 2.2. *Assume that $\beta\hat{\ }1 \prec f$. Since $p_\alpha(s) \geq s$, if it is ever the case that $X^2_{e,s}$ is a proper superset of $\overline{W}_{\alpha,s}$ then we know some ball y in $X^2_{e,s} \cap W_{\alpha,s}$ must later leave $X^2_{e,s}$. Such a ball and a stage will provide us with a Σ^0_1 win for β. So, we can assume that $\overline{X}^2_e$ is a faster enumeration to W_α than the standard enumeration.*

We wait for the next stage $t = p_\beta(s)$ such that $\beta\hat{\ }1 \preceq f_t$. If such a stage does not exist, then $\beta\hat{\ }2 \prec f$, the action of β and all the related δ are finitary, and therefore, Assumption $(\mathcal{A})$ holds.

At this point we have the following

$$X^2_{i,s} \restriction u_{e,s}(x_\delta) = X^2_{e,s} \restriction u_{e,s}(x_\delta) = \overline{W}_{\alpha,s} \restriction u_{e,s}(x_\delta)$$
$$X^2_{e,t'} \restriction u_{e,t'}(x_\delta) \neq \overline{W}_{\alpha,t'} \restriction u_{e,t'}(x_\delta)$$
$$X^2_{e,t} \restriction u_{e,t}(x_\delta) = \overline{W}_{\alpha,t} \restriction u_{e,t}(x_\delta).$$

If $\overline{W}_{\alpha,s} \restriction u_{e,s}(x_\delta) \neq \overline{W}_{\alpha,t'} \restriction u_{e,t'}(x_\delta)$, then some y entered $W_{\alpha,t'}$ after stage s. Then, we have a Σ^0_1 win for δ since $t' < p_\beta(s)$ and $y \in X^2_{i,s}$. So, assume that $\overline{W}_{\alpha,s} \restriction u_{e,s}(x_\delta) = \overline{W}_{\alpha,t'} \restriction u_{e,t'}(x_\delta)$. By the fact that $\beta\hat{\ }1 \prec f$ and Remark 2.2, it must be the case that

$$X^2_{e,t'} \restriction u_{e,t'}(x_\delta) \subsetneq X^2_{e,s} \restriction u_{e,s}(x_\delta).$$

Hence, some ball $y < u_{e,s}(x_\delta)$ must leave $X^2_{e,s}$. Since X^2_e is 2-c.e. that ball y can never return. Hence, since $\beta\hat{\ }1 \prec f$, that ball y must enter W_α and, moreover, it must enter before stage $t = p_\beta(s)$. Therefore, we have a Σ^0_1 win for δ.

Assume that $\beta\hat{\ }1 \prec f$. The infinitely many δ above $\beta\hat{\ }1$ might place infinitely many balls onto the machine. Moreover, we can arrange things such that the set of these balls is not a c.e. set. But at most one of these balls will enter A and Assumption $(\mathcal{A})$ holds.

All that remains at this point is to assign the nodes on the tree such that all the requirements are met. But this can be done in a straightforward fashion. This completes the proof of Theorem 2.1. □

3. Definability and *n*-tardies

We define a property Q_n such that Q_n is nontrivial and if $Q_n(A)$ holds, then A is n-tardy. The property Q_n generalizes Harrington and Soare's property Q first given in Definition 2 of [4] (see also Definition 3.2 of [6]). In particular, property

Q is property Q_2 below. In the next section, we define a nontrivial property $\hat{Q}_n$ using Q_n such that if $\hat{Q}_n(A)$ holds, then A is n-tardy and $\neg\hat{Q}_i(A)$ holds for all $i < n$.

3.1. Q_n

We say $A \sqsubset C$ if there exists a B such that $A \sqcup B = C$, i.e., $A \cup B = C$ and $A \cap B = \emptyset$. We define Q_n separately for n even and odd. We will see in Theorem 3.2 how $Q_n(A)$ corresponds to a game that witnesses the n-tardiness of the set A.

Definition 3.1.

$$(\exists C \supset_m A) \tag{$Q_{2n}(A)$}$$
$$(\forall B_1 \subseteq C)(\forall B_2 \subseteq B_1) \ldots (\forall B_n \subseteq B_{n-1})$$
$$(\exists D_1 \subseteq C)(\exists D_2 \subseteq D_1) \ldots (\exists D_n \subseteq D_{n-1})$$
$$(\forall S \sqsubset C)(\exists T_1 \supseteq \overline{C})(\exists T_2 \subseteq T_1) \ldots (\exists T_n \subseteq T_{n-1})$$
$$\begin{bmatrix} B_1 \cap (S - A) = D_1 \cap (S - A) \\ B_2 \cap (S - A) = D_2 \cap (S - A) \\ \ldots \\ B_n \cap (S - A) = D_n \cap (S - A) \end{bmatrix} \tag{$Q_{2n}(A)$: if}$$
$$\Longrightarrow$$
$$\begin{bmatrix} (A \cup T_2) \cap (S \cap T_1) & = & B_1 \cap (S \cap T_1) \\ (A \cup T_3) \cap (S \cap T_2) & = & B_2 \cap (S \cap T_2) \\ & \ldots & \\ (A \cup T_n) \cap (S \cap T_{n-1}) & = & B_{n-1} \cap (S \cap T_{n-1}) \\ A \cap (S \cap T_n) & = & B_n \cap (S \cap T_n) \end{bmatrix} \tag{$Q_{2n}(A)$: then}$$

$$(\exists Y \subseteq \overline{A}) Q_{2n}(A \cup Y). \tag{$Q_{2n+1}(A)$}$$

Theorem 3.2. *If $Q_n(A)$ holds, then A is n-tardy.*

This is a generalization of Harrington and Soare's Theorem 3.3 in [6] (which in turn is a generalization of their Lemma 1 in [4]) that if $Q(A)$ holds, then A is 2-tardy, and we retain the approach found there. We break this proof into two lemmas, one handling the case where n is even and the other handling the case where n is odd.

Lemma 3.3. *If $Q_{2n}(A)$ implies A is $2n$-tardy for any c.e. set A, then $Q_{2n+1}(A)$ implies A is $2n + 1$-tardy for any c.e. set A.*

Proof. If $Q_{2n+1}(A)$ then $Q_{2n}(A \cup Y)$ holds for some Y disjoint from A. By assumption $A \cup Y$ is $2n$-tardy. Thus, if $p(s)$ is a total computable function, there is some $2n$-c.e. set $X^{2n} = (X_1 - X_2) \cup \cdots (X_{2n-1} - X_{2n})$ equal to $\overline{A \cup Y}$ such that $x \in X^{2n}_s \implies x \notin A_{p(s)}$. Let

$$X^{2n+1} = X^{2n} \cup Y = (X_1 - X_2) \cup \cdots \cup (X_{2n-1} - X_{2n}) \cup Y.$$

Since $Y \cap A = \emptyset$, $X^{2n+1} = \overline{A}$ and $x \in X^{2n+1}_s \implies x \notin A_{p(s)}$. Since $p(s)$ was arbitrary, A is $2n + 1$-tardy. □

Lemma 3.4. *$Q_{2n}(A)$ implies A is $2n$-tardy.*

Proof. Fix A and C (and indexes for them) such that A satisfies $Q_{2n}(A)$ via C and $A \subseteq C \searrow A$ where the latter property can be guaranteed purely by change of index. Following the approach in [4,6], we think of $Q_{2n}(A)$ as a two player game between the $\exists$-player (called EXISTS here and RED in [6]) who plays the sets $\vec{D} = (D_1, D_2, \ldots, D_n)$ and $\vec{T} = (T_1, \ldots, T_n)$ and the $\forall$-player (called FORALL here and BLUE in [6]) who plays the sets $B_1, B_2, \ldots, B_n$ and $S \sqsubset C$. Should $A, C, S, \vec{D}, \vec{T}$ witness the satisfaction of $Q_{2n}(A)$ we say the EXISTS player wins. Otherwise, the FORALL player wins. Since C witnesses the satisfaction of $Q_{2n}(A)$, the EXISTS player must have a winning strategy. Given any total computable function $p(s)$, the proof will proceed by specifying a strategy for the FORALL player such that winning response $\vec{D}, \vec{T}$ of the EXISTS player allows us to build a $2n$-c.e. set X^{2n} witnessing that A is $2n$-tardy.

Given a total computable function $p(s)$, FORALL will respond by building $\vec{B}$. However, in the construction of B, FORALL will want to use information about the particular sets $\vec{D}, \vec{T}$ played by EXISTS, but $\vec{B}$ must be built without knowledge of $\vec{D}$ or $\vec{T}$. We let $\vec{B}$ react to the particular choice of $\vec{D}$ by simultaneously building $\vec{B}$ and a sequence of sets $S_e \sqsubset C$ such that on S_e, the collection $\vec{B}$ plays against $\vec{D}_e = (W_{e_1}, \ldots, W_{e_n})$. During this construction, $\vec{B}$ will be built so that, for every e, property ($Q_{2n}(A)$: if) holds for $S = S_e, \vec{D} = \vec{D}_e$. Thus, for EXISTS to have a winning strategy, there must be some $\vec{T}$ witnessing the satisfaction of ($Q_{2n}(A)$: then).

We now further divide up the sets S_e into the sets $S_{e,j}$ with $S_e = \bigsqcup_{j\in\omega} S_{e,j}$ so that FORALL builds $\vec{B}$ to play against $\vec{D}_e, \vec{T}_j$ on $S_{e,j}$. Since S must be played without knowledge of $\vec{T}$, we appear to run the risk that the winning strategy for EXISTS never plays $\vec{T}_j$ against $S_{e,j}$. However, since $\vec{B}, \vec{D}_e, S_e$ satisfies ($Q_{2n}(A)$: if), there is some j such that $\vec{B}, \vec{T}_j, S_e$ satisfy ($Q_{2n}(A)$: then). But as $S_{e,j} \subset S_e$, it follows that $\vec{B}, \vec{T}_j, S_{e,j}$ satisfy ($Q_{2n}(A)$: then). Thus, provided for all e we maintain ($Q_{2n}(A)$: if) for $S_e, \vec{D}_e$, we may assume that for some e, j the sets $\vec{B}, \vec{D}_e, \vec{T}_j, S_{e,j}$ satisfy both ($Q_{2n}(A)$: if) and ($Q_{2n}(A)$: then).

We let α range over indexes e, j for n tuples of c.e. sets and define

$$\begin{aligned} S_\alpha &= S_{e,j} \\ D_i^\alpha &= W_{e_i} \\ T_i^\alpha &= W_{j_i} \end{aligned}$$

where we stipulate that our indexes satisfy

$$\begin{aligned} D_1^\alpha &\subseteq C \\ D_{i+1}^\alpha &\subseteq D_i^\alpha \\ T_{i+1}^\alpha &\subseteq T_i^\alpha. \end{aligned}$$

Relative to a particular choice of $\vec{B}$, the predicate $F(\alpha)$ asserting that the sets $\vec{B}, \vec{D}_e, \vec{T}_j, S_{e,j}$ satisfy both ($Q_{2n}(A)$: if) and ($Q_{2n}(A)$: then) is Π_2^0. Thus, there is a uniformly computable sequence of predicates $F_s(\alpha)$ referring only to the commitments we have made about $\vec{B}$ by stage s in our construction such that $F(\alpha) \leftrightarrow (\exists^\infty s)\, F_s(\alpha)$. Using this predicate, we define a strong array of finite sets U_i^α for every α and $i \in [1, n]$ as follows.

$$\begin{aligned} x \in U_{1,s}^\alpha &\iff x \in U_{1,s-1}^\alpha \vee [\, s \geq x \wedge x \in (T_{1,s}^\alpha - C_s) \wedge F_s(\alpha)\,]. \\ x \in U_{i,s}^\alpha &\iff x \in U_{i-1,s}^\alpha \cap T_{i,s}^\alpha. \end{aligned}$$

By way of the Slowdown Lemma ([9] p. 284) applied to the above arrays, we define

$$X_{2i-1}^\alpha = \bigcup_{s\in\omega} U_{i,s}^\alpha$$

satisfying

$$X_{2i-1,s}^\alpha \subset U_{i,s}^\alpha.$$

If we build $S_{e,j}$ as described, there must be some least α for which $F(\alpha)$ holds by the remarks above. For that α, $U_1^\alpha \supset \overline{C}$ since ($Q_{2n}(A)$) requires that $T_1^\alpha \supset \overline{C}$ and $F_s(\alpha)$ holds for infinitely many s. Hence, $X_1^\alpha \supset^* \overline{A}$ since $A \subset_m C$. We also have $T_{i+1} \subseteq T_i$ and $X_{2i-1}^\alpha = T_i^\alpha \cap X_1^\alpha$ by definition. So, if the sequence $\vec{T^\alpha}$ witnesses that ($Q_{2n}(A)$: then) holds, we may replace each T_i^α with X_{2i-1}^α without falsifying ($Q_{2n}(A)$: then).

We now build S_α with the intention that (with finitely many exceptions) every element that is in $X_1^\alpha \cap A$ is in S_α. If $x \in C_{s+1} - C_s$, take the least α such that $x \in U_{1,s}^\alpha$ and enumerate x into S_α. If no such α exists, enumerate x into the garbage set S_{-1}. Note that $C = \bigsqcup_{\alpha\in 2^{<\omega}} S_\alpha \sqcup S_{-1}$ by construction, so, $S_\alpha \sqsubset C$ for every α. Furthermore, by construction, once x enters C it can no longer enter U_1^α for any α. Suppose α is the least such that $F(\alpha)$ holds. Since U_1^β is finite for every $\beta < \alpha$, we have $U_1^\alpha \cap C \subset^* S_\alpha$. Hence, for all $i \in [1, n]$

$$X_{2i-1}^\alpha \cap C \subset^* S_\alpha. \tag{3.5}$$

Conversely, $S_\alpha \subseteq X_1^\alpha$. We are now ready to define $\vec{B}$ and the even components of X^α. Let

$$X_{2i}^\alpha = S_\alpha \cap D_i^\alpha$$

where by way of the Slowdown Lemma ([9] p. 284), we ensure that

$$X_{2i+2}^\alpha \subseteq X_{2i}^\alpha \searrow X_{2i+2}^\alpha.$$

Since $X_1^\alpha \cap C \subset^* S_\alpha$ and $X_1^\alpha \cap \overline{C} \subseteq \overline{A}$, requiring X_{2i}^α to be a subset of S_α is no handicap to ensuring $X^\alpha = \overline{A}$. If $F(\alpha)$ holds, then we claim that

$$\begin{aligned} &X_{2j-1}^\alpha \cap X_{2j}^\alpha \cap \overline{A} \subseteq X_{2j+1}^\alpha \\ &X_{2n-1}^\alpha \cap X_{2n}^\alpha \cap \overline{A} = \emptyset. \end{aligned} \tag{3.6}$$

To see this, let $x \in X_{2j-1}^\alpha \cap X_{2j}^\alpha \cap \overline{A}$. Since $X_{2j}^\alpha = D_j^\alpha \cap S_\alpha$, we have $x \in D_j^\alpha \cap (S_\alpha - A)$ which by ($Q_{2n}(A)$: if) is contained in B_j. By a prior remark, we may substitute X_{2j-1}^α in for T_j in ($Q_{2n}(A)$: then), and since $x \in B_j \cap S_\alpha \cap X_{2j-1}^\alpha$, we have $x \in A \cup X_{2j+1}^\alpha$.

Since $x \notin A$, we have $x \in X^\alpha_{2j+1}$. Moreover, by similar reasoning, $X^\alpha_{2n-1} \cap X^\alpha_{2n} \cap \overline{A} = \emptyset$. We then derive the following containments.

$$\begin{aligned} \overline{A} \cap S_\alpha &\subseteq X^\alpha \\ \overline{A} &\subseteq^* X^\alpha. \end{aligned} \tag{3.7}$$

For the first containment, if $x \in \overline{A} \cap S_\alpha$ then, as $S_\alpha \subseteq X^\alpha_1$, there is a maximal j such that $x \in X^\alpha_{2j-1}$. Since the even indexed components of X^α are nested, if $x \notin X^\alpha_{2j}$ then $x \in X^\alpha$, and we are done. If $x \in X^\alpha_{2j}$, then (3.6) yields a contradiction. The second containment follows since $X^\alpha \supseteq X^\alpha_1 - S_\alpha$ (by definition, each $X^\alpha_{2j} \subseteq S_\alpha$, so no elements outside of S_α are removed from X^α) and $X^\alpha_1 \supseteq^* \overline{A}$. We now define $\vec{B}$ so that the other direction of containment and the tardiness property hold.

$$x \in B_i \iff (\exists \alpha)(\exists s)\left[x \in X^\alpha_{2i,s} \wedge x \notin A_{p(s)}\right]. \tag{3.8}$$

Tracing out the definition of X^α_{2i}, it is evident that on $S_\alpha - A$ we have $B_i = D^\alpha_i$. Hence, by our earlier arguments, there is some α such that $F(\alpha)$ holds. Now let α be the least such. Since $B_i \cap S_\alpha \subseteq D^\alpha_i \cap S_\alpha$, using ($Q_{2n}(A)$: then) we see

$$A \cap X^\alpha_{2i-1} \cap S_\alpha \subseteq B_i \cap X^\alpha_{2i-1} \cap S_\alpha \subseteq D^\alpha_i \cap S_\alpha = X^\alpha_{2i}.$$

Thus, if $x \in A \cap S_\alpha$ then $x \notin X^\alpha$. By (3.5), $X^\alpha \cap C \subseteq^* S_\alpha$. Since $X^\alpha \cap C \subseteq^* S_\alpha$ and $A \subseteq C$, this entails $\overline{A} \supseteq^* X^\alpha$. Putting this together with (3.7), we conclude

$$\begin{aligned} \overline{A} \cap S_\alpha &= X^\alpha \cap S_\alpha \\ \overline{A} &=^* X^\alpha. \end{aligned}$$

We now argue that X^α has the desired tardiness properties. Suppose $x \in X^\alpha_1$ and $x \in A \cap S_\alpha$. Let j be the greatest such that $x \in X^\alpha_{2j-1}$. Now suppose x enters X^α_{2j} at stage s. If $x \in A_{p(s)}$ then by (3.8) $x \notin B_i$. But as $x \in X^\alpha_{2j-1} \cap S_\alpha$, it follows from ($Q_{2n}(A)$: then) that $x \notin A$. This is a contradiction. Therefore,

$$x \in S_\alpha \cap X^\alpha_s \implies x \notin A_{p(s)}.$$

Now set $X_{2i} = X^\alpha_{2i}$ and $X_{2i-1} =^* X^\alpha_{2i-1}$ where we build X_{2i-1} by removing the finitely many members of $A \cap \bar{S_\alpha} \cap X^\alpha_1$ from X^α_{2i-1} and adding the finitely many members of $\overline{A} - X^\alpha_1$ so that the $2n$-c.e. set X defined by these X_i equals $\overline{A}$. The set X witnesses that A is $2n$-tardy with respect to $p(s)$. Since $p(s)$ was arbitrary, we can conclude A is $2n$-tardy. □

Taken together these lemmas suffice to establish Theorem 3.2.

4. Proper satisfaction Q_n

By Theorem 3.2, we have a countable collection of properties Q_n for $n \geq 2$ that are preserved under automorphism and guarantee incompleteness. It is easily verified that $Q_{2n}(A)$ implies $Q_{2n+2}(A)$ so to illustrate the existence of countably many incomplete orbits, we must show this hierarchy of properties does not collapse. In particular, it suffices to show that for every $n > 2$ there is a properly n-tardy A satisfying $Q_n(A)$. We then define

$$\hat{Q}_n(A) \iff Q_n(A) \wedge \neg Q_{n-1}(A) \wedge \cdots \wedge \neg Q_2(A).$$

By constructing A to be a *properly* n-tardy set satisfying $Q_n(A)$ we guarantee A satisfies $\hat{Q}_n(A)$ because, by Theorem 3.2, any set satisfying $Q_m(A)$ for $m < n$ must be m-tardy. Since the properties $\hat{Q}_n$ are pairwise incompatible it suffices to produce such an A for every $n \geq 2$ to demonstrate the existence of countably many disjoint orbits.

Theorem 4.1. *For all $m \geq 2$ there is a properly m-tardy A satisfying $Q_m(A)$. Therefore there are countably many disjoint incomplete orbits of the c.e. degrees under $\subset$.*

Theorem 4.1 generalizes Harrington and Soare's construction of a set that satisfies property $Q = Q_2$ in ([4], Lemma 2). To build a set A that satisfies Q_n, we build a major superset C of A that provides early warning about which elements may enter A, following Harrington and Soare's approach. Harrington and Soare ([6] Corollary 3.10) later demonstrated that the property $Q = Q_2$ holds of any small major 2-tardy subset of some C. We leave open the question of whether some similar property suffices to guarantee Q_n holds.

As in Theorem 3.2, we consider the even and odd cases separately for Theorem 4.1. We first show that there is a properly $2n$-tardy satisfying Q_{2n} and then modify this argument to yield a properly $2n+1$-tardy satisfying Q_{2n+1}. In the next lemma, we construct the sets A and C with $A \subset_m C$, $A = C \searrow A$, and other dynamic properties. We then show that A as constructed satisfies Q_{2n} in Section 4.3. The remaining properties in Lemma 4.2 prescribe the order that elements pass through A, C, and the $2n$-c.e. approximations of $\overline{A}$ and how these sets are nested. These properties are less crucial for showing $Q = Q_2$ is satisfied, since in that case only 2-c.e. approximations of $\overline{A}$ are being considered. We believe that any proof of satisfaction for the arbitrary Q_n will need to establish the existence of sets satisfying something like Lemma 4.2.

4.1. Building a properly $2n$*-tardy* A *and* C

Lemma 4.2. *For every* $n \geq 1$ *there is a properly* $2n$*-tardy set* A *and a c.e. set* C *with* $C \supset_m A$ *such that, for every total computable nondecreasing function* p*, there is a* $2n$*-c.e. set* X_e^{2n} *satisfying*

$$\overline{A} = X_e^{2n} = (X_{e_1}^{2n} - X_{e_2}^{2n}) \cup \cdots \cup (X_{e_{2n-1}}^{2n} - X_{e_{2n}}^{2n}) \tag{4.3a}$$

$$(\forall x)(\forall s)\left(x \in X_{e,s}^{2n} \implies x \notin A_{p(s)}\right) \tag{4.3b}$$

$$(\forall k < 2n)\left[k > 0 \implies X_{e_{k+1}}^{2n} = C \searrow X_{e_{k+1}}^{2n}\right] \tag{4.3c}$$

$$X_{e_1}^{2n} \supseteq X_{e_2}^{2n} \supseteq \ldots \supseteq X_{e_{2n-1}}^{2n} \supseteq X_{e_{2n}}^{2n} \tag{4.3d}$$

$$(\forall i < 2n)[X_{e_{i+1}}^{2n} = X_{e_i}^{2n} \searrow X_{e_{i+1}}^{2n}]. \tag{4.3e}$$

Proof of Lemma 4.2. We start with a simple set $\hat{C}$ satisfying $|\hat{C} \restriction_{2x}| \leq x$ and simultaneously construct $C \supseteq \hat{C}$ (so C is simple as well) and A. During the construction, we refer to the index of C as a c.e. set so we can measure its speed of enumeration. We justify this circularity by regarding the construction as a computable function operating on a guess at the index for C and returning an index for the resulting C. We then apply the recursion theorem to obtain C.

To build A, we will work to meet the requirements $\mathcal{N}_e$, $\mathcal{M}_e$, $\mathcal{R}_e$ specified below to which we assign priorities $3e$, $3e+1$, $3e+2$, respectively. These requirements are thought of as being laid out vertically in order of priority. Ultimately, the true construction will take the form of a tree argument in the style of Theorem 2.1. Rather than repeat the standard details of the tree layout, we instead present the argument as if it were a infinite injury pinball argument with A at the bottom of the machine and the requirements stretching upwards. As in Theorem 2.1, balls (numbers) will be released at requirements of the form $\mathcal{M}_e$ and $\mathcal{R}_e$. These balls attempt to flow down through the negative requirements $\mathcal{N}_e$ below. When (and if) they reach the bottom, they are enumerated into A. Ultimately, however, we will observe that the computable corrections required by infinite injury can simply be considered as the action of the tree when phrased as a Π_2^0 tree argument and can thus be squared with requirements (4.3d) and (4.3e). We may insist that (4.3c) holds by pausing the construction until elements appear in the canonical enumeration of C as necessary.

4.1.1. The $\mathcal{N}_e$ *module (*A *is* $2n$*-tardy)*

To show A is $2n$-tardy, we meet the following requirements.

$$\text{If } \varphi_e \text{ total \& nondecreasing, then } \left(\exists X_e^{2n}\right)\left(X_e^{2n} = \overline{A} \wedge (\forall x)(\forall s)\left[x \in X_{e,s}^{2n} \implies x \notin A_{\varphi_e(s)}\right]\right). \tag{$\mathcal{N}_e$}$$

We act to meet this requirement as follows. At the start of stage $s > e$, let l to be maximal such that $(\forall x < l)[\varphi_{e,s}(x)\downarrow]$ and put every $x < l$ into $X_{e_1}^{2n}$ that is not already in A or located below $\mathcal{N}_e$ along our list of requirements. If a ball x targeted for A by a lower priority requirement reaches $\mathcal{N}_e$ at stage s and it is not yet in $X_{e_1}^{2n}$, it is immediately allowed to fall through to the next negative requirement along the path to A. Otherwise, if $x \in X_{e_1}^{2n}$ let j be the largest index such that $x \in X_{e_j}$ (such an index will be even by construction). Place x into $X_{e_{j+1}}^{2n+1}$ and delay x from passing through to the next highest priority $\mathcal{N}_e$ until the first stage t such that $\varphi_{e,t}(s)\downarrow$ is reached. If a lower priority requirement cancels its attempt to place some $x \in X_{e_{2j}}^{2n} - X_{e_{2j+1}}^{2n}$ into A before x enters A, the element x is placed into $X_{e_{2j+1}}^{2n}$. Observe that if φ_e is partial, $X_{e_1}^{2n}$ will be finite and only finitely many balls will be permanently delayed by $\mathcal{N}_e$.

This action suffices to meet $\mathcal{N}_e$ modulo the balls put into A by higher priority requirements. At the end of the construction, we will observe that the set of such elements is computable. Thus, we can modify X_e^{2n} to satisfy the requirement without sacrificing any of the desired properties.

4.1.2. The $\mathcal{M}_e$ *module (*$A \subset_m C$*)*

To ensure that $C \supseteq_m A$, we satisfy the following requirements.

$$W_e \supseteq \overline{C} \implies W_e \supseteq^* \overline{A}. \tag{$\mathcal{M}_e$}$$

We construct $A \subset_m C$ by ensuring that if $W_e \supset \overline{C}$ then $C - A \subseteq^* W_e$. If we knew from the outset that $W_e \cap C$ was infinite and $W_e \supseteq \overline{C}$, we could ensure $C - A \subseteq^* W_e$ by enumerating elements in C but not W_e into A. We construct C to be simple so that $W_e \cap C$ is infinite if $W_e \supseteq \overline{C}$. We cannot, however, determine effectively whether $W_e \supseteq \overline{C}$ so we instead assume that we have seen the entirety of W_e and correct our construction if more elements enter W_e. In particular, if W_e extends to contain $\overline{C} \cap [0, l]$, we respond by enumerating (almost) every $x < l$ with $x \in C$ but not yet in W_e into A to keep $C - A \subseteq^* W_e$. To ensure $C - A$ is infinite, we absolve the first e (candidate) members of $C - A$ from being affected by $\mathcal{M}_e$.

We produce the sets A and C with $C \supset_m A$ by combining the standard construction of a major subset with both positive and negative tardiness requirements. We fix $\{C_s\}_{s \in \omega}$, a stagewise approximation to C, such that C_0 is some infinite computable subset of $\hat{C}$ and other elements enter C_s only when they are enumerated into $\hat{C}$ or placed into C by $\mathcal{R}_e$, as described below. We also fix a countable collection of markers m_k shared across all the $\mathcal{M}_e$ requirements whose position at stage s we denote by $m_{k,s}$ with the intention (which we almost fulfill) of letting them come to rest on $C - A$. We describe the motion of these markers in terms of an e-state construction. Instead of maximizing the e-state of our markers, which would guarantee that any c.e. set containing infinitely many elements from $C - A$ contains almost all of them, we only maximize the e-state for c.e. sets threatening to contain $\overline{C}$. To this end, we employ the following twist on the notion of an e-state. This adjustment reduces the number of times when we must pull markers to increase the e-state.

Definition 4.4. The *C-complementing e-state of x*, denoted $\epsilon_e^C(x)$, is defined to be the $\nu \in 2^{e-1}$ such that

$$(\forall i < e)\left(\nu(i) = 1 \leftrightarrow W_i \supseteq (\overline{C} \cap [0, x]) \wedge x \in W_i\right).$$

The *C-complementing e-state of x at stage s*, denoted $\epsilon_e^C(x, s)$, is defined to be the $\nu \in 2^{e-1}$ such that

$$(\forall i < e)\left(\nu(i) = 1 \leftrightarrow W_{i,s} \supseteq (\overline{C_s} \cap [0, x]) \wedge x \in W_{i,s}\right).$$

At the start of the construction, we place m_k on the kth element of C_0. At the start of stage $s+1$ for every $x \in C_{s+1} - C_s$, we pick the least k such that $m_{k,s} > x$, define $m_{k,s+1} = x$, and shift the markers after m_k down to their predecessors' location. Note that since $m_{k,s} > x$ if $\epsilon_e^C(m_{k,s}, s)(i) = 1$ then since $x \notin C_s$ it follows that $x \in W_{i,s}$ and thus $\epsilon_e^C(m_{k,s+1}, s+1) \geq_L \epsilon_e^C(m_{k,s}, s)$ where $\geq_L$ denotes the lexicographic order.

After all the requirements with greater priority than $\mathcal{M}_e$ have acted at stage s, we search for the least k, k' with $k' > k \geq e$ and

$$\epsilon_{e+1}^C(m_{k',s}, s) >_L \epsilon_{e+1}^C(m_{k,s}, s). \tag{4.5}$$

We then move the marker m_k to the location occupied by $m_{k'}$. We shift the later markers up accordingly and target the locations previously occupied by m_j for $k \leq j < k'$ for entry into A.

We inductively argue that each marker comes to rest. Pick s large enough so that every $m_{k'}$ for $k' < k$ has already come to rest on its final position and then choose $s' > s$ so that $\epsilon_{k+1}^C(m_{k,s'}, s')$ is maximal for $s' \geq s$. The marker m_k cannot be moved at this point unless new elements are enumerated into C. By the above remarks, this movement cannot decrease $\epsilon_{k+1}^C(m_{k,s'}, s')$. Eventually, no further elements of C are enumerated below m_k, and the marker m_k comes to rest.

We now argue that $\mathcal{M}_e$ is satisfied if all but finitely many of the elements targeted for A by $\mathcal{M}_e$ eventually enter A. To see this, fix some $W_e \supseteq \overline{C}$ and note that the intersection of all sets W_i for $i \leq e$ such that $W_i \supseteq \overline{C}$ also contains $\overline{C}$. By the simplicity of C, this intersection must have an infinite intersection with C. It follows that all but finitely many elements of $C - A$ are contained in W_e and, indeed, all but finitely many elements in $C - A$ have some C-complementing $(e+1)$-state ν. Moreover, those elements targeted for A by $\mathcal{M}_e$ form a computable set as, for large enough x, $\mathcal{M}_e$ targets x for A only if it has done so by the time we see a marker above x attain the C-complementing e-state ν.

4.1.3. The $\mathcal{R}_e$ module (A properly $2n$-tardy)

Lastly, we must guarantee that A is not $(2n-1)$-tardy. To that end, we ensure that no $(2n-1)$-c.e. set witnesses that a particular enumeration of A is $(2n-1)$-tardy with respect to a certain nondecreasing computable function. For a c.e. set $B = W_i$ given in the canonical enumeration, we define $p^B(s)$ so that if the construction places x into A at stage s then $x \in B_{p^B(s)}$. If $B = A$ then $p^B(s)$ will be total. Since the requirements for $B = W_i$ do not interact significantly with those requirements for $B = W_{i'}$, we drop the subscript i from the statement of the requirement.

$$(\exists x)\begin{bmatrix} (\exists s)[x \in Y_{e,s}^{2n-1} \wedge x \in (A_s - A_{s-1})] \\ \vee \\ Y_e^{2n-1}(x) \neq \overline{A}(x) \vee A \neq B \end{bmatrix}. \tag{$\mathcal{R}_e$}$$

Suppose $B = A$ and $\mathcal{R}_e$ is satisfied. If $Y_e^{2n-1} = \overline{A}(x) = \bar{B}(x)$, then there is a stage s so that $x \in Y_{e,s}^{2n-1}$ and $x \in A_s - A_{s-1}$. But then $x \in B_{p^B(s)}$ by definition of p^B. Hence, $\mathcal{R}_e$ guarantees that Y_e^{2n-1} does not witness that $B = A$ is $2n-1$-tardy with respect to p^B.

If $\mathcal{R}_e$ appears unsatisfied, we attempt to hold a ball x out of A until x enters Y_e^{2n-1}. (If x never enters Y_e^{2n-1}, we have $Y_e^{2n-1}(x) \neq \overline{A}(x)$.) We then target x for entry into A on behalf of $\mathcal{R}_e$, and if x enters A before leaving Y_e^{2n-1}, then $\mathcal{R}_e$ is satisfied. If instead x leaves Y_e^{2n-1} before x enters A, then we cancel the attempt to place x into A. Moreover, we return x to $\mathcal{R}_e$ and again hold x out of A until x enters Y_e^{2n-1}. Since Y_e^{2n-1} can change on x one fewer time than any $2n$-c.e. approximation to $\overline{A}$, either there is a stage s such that $x \in Y_{e,s}^{2n-1}$ and $x \in A_s - A_{s-1}$ or x enters the $(2n-1)$th component of Y_e^{2n-1}. In the latter case, we place x into A since then $\bar{B} \neq Y_e^{2n-1}$ if $B = A$. Note that we only place an element into A after placing it in C (if it is not already in C). The real complexity in meeting $\mathcal{R}_e$ is guaranteeing that we can reserve some x large enough so that x is neither permanently restrained by some higher priority $\mathcal{N}_{e'}$ nor placed into A by some higher priority $\mathcal{M}_{e'}$. We now discuss how such an x is obtained.

During the construction, each $\mathcal{R}_e$ will reserve a finite collection of intervals $\left\{\left[l_i^e, h_i^e\right]\right\}_{i=1}^{n_e}$ for its exclusive use such that if $e \neq e'$ or $i \neq i'$ then $\left[l_i^e, h_i^e\right]$ and $\left[l_{i'}^{e'}, h_{i'}^{e'}\right]$ are disjoint. Inside each interval, $\mathcal{R}_e$ will maintain a marker r_i^e with position $r_{i,s}^e$ at stage s to indicate the currently active element in that interval. Only $\mathcal{R}_e$ or a higher priority $\mathcal{M}_{e'}$ is allowed to target a member of $\left[l_i^e, h_i^e\right]$ for A. Whenever any marker r_i^e has been targeted for A but delayed by some higher priority $\mathcal{N}_{e'}$, we reserve another interval for $\mathcal{R}_e$. If $\mathcal{R}_e$ already has $j-1$ intervals we select l_j^e to be the first element larger than every previously defined interval for any requirement. We then select h_j^e to be the least number currently occupied by some marker m_k such that $2(l_j^e + 2^e + 1) < h_j^e$ and place r_j^e on h_j^e. Since $|\hat{C} \restriction_{2x}| \leq x$, there are at least 2^e elements in $\left[l_j^e, h_j^e\right]$ that will not be enumerated into $\hat{C}$, leaving $\mathcal{R}_e$ and higher priority $\mathcal{N}_{e'}$ complete control over placing these elements into C.

If the first clause in ($\mathcal{R}_e$) is not yet satisfied, we act if some r_j^e occupies x and either x is not currently targeted to enter A but $x \in Y_{e,s}^{2n-1}$ or x is currently targeted to enter A but $x \notin Y_{e,s}^{2n-1}$. In the former case, we target x for entry into A and in the latter case, we cancel our targeting of x for A (placing x into those $X_{e'}^{2n}$ being built at higher priority $\mathcal{N}_{e'}$). If at some stage s, element $r_{i,s}^e$ is targeted for A by a higher priority $\mathcal{M}_{e'}$, then set $r_{i,s+1}^e$ to the largest $x < r_{i,s}^e$ with $x \notin C_s$ and enumerate x into C. (We show below such an x exists in the reserved interval).

We argue that each $\mathcal{R}_e$ only reserves finitely many elements and is eventually satisfied. Note that we only move r_i^e at stage s if there is some element $y > r_{i,s}^e$ with $\epsilon_{e+1}^C(y, s) >_L \epsilon_{e+1}^C(r_{i,s}^e, s)$. By enumerating $r_{i,s+1}^e$ into C, we cause the marker m_k occupying the least $y' > r_{i,s+1}^e$ to be shifted down to $r_{i,s+1}^e$. By our remarks in (4.1.2), we know that $\epsilon_{e+1}^C(r_{i,s+1}^e, s+1) \geq_L \epsilon_{e+1}^C(y, s)$. Combining these inequalities, we see that $\epsilon_{e+1}^C(r_{i,s+1}^e, s+1) >_L \epsilon_{e+1}^C(r_{i,s}^e, s)$. Since there are only 2^e many C-complementing $(e+1)$-states, we can move r_i^e at most $2^e - 1$ times. By choice of h_i^e, we know that each time we can find some element in $[l_i^e, h_i^e]$ not yet in C_s. Hence, r_i^e eventually occupies a location that is not targeted for A by a higher priority $\mathcal{M}_{e'}$. Now, if $\mathcal{R}_e$ reserves only finitely many intervals, it is satisfied, so assume it reserves infinitely many intervals. In this case, let $[l_i^e, h_i^e]$ be an interval with l_i^e so large that no element in this interval is permanently restrained by any $\mathcal{N}_{e'}$ for $e' \leq e$, and let x be the location r_i^e settles upon. But now the strategy for $\mathcal{R}_e$ ensures the element x will witness a victory against Y_e^{2n-1} as described above and no more intervals will be reserved for $\mathcal{R}_e$.

This completes the construction of A and C.

4.1.4. Verification of Lemma 4.2

We now need to verify that A and C have the claimed properties. If φ_e is total then eventually every element in $\overline{A}$ enters X_e^{2n} or remains stalled at some $\mathcal{N}_{e'}$ with $e' < e$ and $\mathrm{dom}\varphi_{e'}$ finite. Thus, by adding these finitely many stalled balls to $X_{e_j}^{2n}$ for j the least odd number so that $x \notin X_{e_j}^{2n}$, we can assume that X_e^{2n} contains $\overline{A}$. Moreover, we can make this finite adjustment without disrupting the property that balls enter the earlier components of X_e^{2n} before the later ones and only enter $X_{e_2}^{2n}$ after C. Conversely, X_e^{2n} is contained in the union of $\overline{A}$ and the set of elements placed into A by requirements $\mathcal{R}_{e'}$ or $\mathcal{M}_{e'}$ for $e' < e$, which in the former case is a finite set and the latter a computable set. Thus, X_e^{2n} is contained in the union of $\overline{A}$ and a computable subset R of A so we can fix X_e^{2n} to be equal to $\overline{A}$ by intersecting $\overline{R}$ with every positive (odd) component of X_e^{2n}. Since we do not alter the even components of X_e^{2n}, we do not slow down any elements from leaving X_e^{2n}, and so retain the required tardiness property. But, by taking elements out of the odd components but not the even ones, we may now violate (4.3d) and (4.3e).

However, the need to adjust X_e^{2n} after the construction is really only a consequence of our decision to cast the construction as a pinball argument for ease of presentation rather than a Π_2^0 tree construction. By performing this construction in the same fashion as that in Theorem 2.1, our ad hoc modification of X_e^{2n} becomes unnecessary as nodes γ for requirements $\mathcal{N}_{e'}$ can simply delay adding balls to the components of $X_{e'}^{2n}$ until every higher priority requirement $\mathcal{R}_e$ that γ guesses will act infinitely often believes it will not need to add that ball to $X_{e'}^{2n}$. The node γ can simply reset its construction of $X_{e'}^{2n}$ whenever a requirement $\mathcal{R}_e$, which γ believes only acts finitely many times, acts. Understood in terms of the tree construction, the reservation of balls by $\mathcal{R}_e$ acting at $\mathcal{N}_{e'}$ simply becomes the constraint that any nodes above or to the right of γ cannot pick these elements as new balls. $\square$

4.2. Building a properly $(2n+1)$-tardy A and C

We can prove similarly a version of Lemma 4.2 for the odd case.

Lemma 4.6. *For every $n \geq 1$, there is a properly $2n+1$-tardy set A, a c.e. set Z disjoint from A, so that $\hat{A} = A \sqcup Z$ satisfies the conditions of Lemma 4.2. In particular, $\hat{A}$ is $2n$-tardy and satisfies all the demands on the enumeration order.*

Proof. In order to ensure that A is *properly* $2n+1$-tardy, we dynamically build $\hat{A}$ as in Lemma 4.2 and we decide whether to put x into A or Z only once we have made an irrevocable commitment to place x into $\hat{A}$. Specifically, to show that A is $2n+1$-tardy, it suffices that $\hat{A}$ satisfies $\mathcal{N}_e$ and $\mathcal{M}_e$ as in Lemma 4.2 (i.e., $\hat{A}$ is $2n$-tardy with $\hat{A} \subset_m C$). To show that A is not $2n$-tardy, we show A satisfies $\mathcal{R}_e$ except now we diagonalize against $2n$-c.e. sets Y_e^{2n}.

If we put x into $\hat{A}$ as the result of $\mathcal{M}_e$, we place x into A. Then, $\mathcal{M}_e$ will be satisfied in the same manner as before. We now focus on those balls placed in $\hat{A}$ by some requirement $\mathcal{R}_e$. We place into A any balls that were enumerated into $\hat{A}$ by some $\mathcal{R}_e$ before entering the $2n$th component of $X^{2n}_{e',s}$ for higher priority requirements $\mathcal{N}_{e'}$. These balls entered $\hat{A}$ to obtain an immediate victory for $\mathcal{R}_e$ by showing that either $A \neq B$ or that x does not leave $Y^{2n}_{e,s}$ soon enough before entering B. By placing these balls into A, we ensure that if $B = A$, then $Y^{2n}_{e,s}$ does not witness that A is $2n$-tardy with respect to p^B. This leaves the case where x enters the $2n$th component of $X^{2n}_{e',s}$ for some higher priority requirement $\mathcal{N}_{e'}$. This only occurs if x enters the $(2n-1)$th component of $Y^{2n}_{e,s}$ and the construction of $\hat{A}$ responds by targeting x for $\hat{A}$ by placing it in the sets $X^{2n}_{e'_{2n}}$ for higher priority requirements $\mathcal{N}_{e'}$. As far as the construction of $\hat{A}$ is concerned, once x has entered $X^{2n}_{e_{2n}}$, it must enter $\hat{A}$ (modulo finite injury). However, when x reaches the root, we check if x is still in $Y^{2n}_{e,s}$. If so, we place x into A for the immediate victory. If $x \notin Y^{2n}_{e,s}$, we place x into Z so that if $B = A$, then $Y^{2n}_e \neq \overline{A}$. Hence, $A = \hat{A} - Z$ is properly $2n+1$-tardy.

Since the only elements entering $\hat{A}$ but not A pass through all the intermediate components $X^{2n}_{e'_k}$ in order at the higher priority $\mathcal{N}_{e'}$ below $\mathcal{R}_e$, the ordering properties trivially hold at these nodes. At the remaining nodes, x may have become stuck in $X^{2n}_{j_2}$ or some other component. However, this concern is easily addressed by taking any balls we place into Z and slowly running them through the components of X^{2n}_j in order. Then, we can use a slower enumeration of Z as the final component to X^{2n}_j, making a $2n+1$-c.e. set that satisfies the hypotheses of the theorem. $\square$

4.3. Proof of Theorem 4.1

We now prove Theorem 4.1, i.e., we show that the above sets A in Lemmas 4.2 and 4.6 satisfy $Q_m(A)$. Since the definition of $Q_{2n+1}(A)$ is simply $(\exists Z \subseteq \overline{A})Q_{2n}(A \cup Z)$ and $\hat{A} = A \sqcup Z$ satisfies the conditions of Lemma 4.2, we simply need to show that $Q_{2n}(A)$ holds for A as in Lemma 4.2. Our proof of satisfaction is based on and generalizes Lemma 3.6 in [6]. We also spell out some details left to the reader in that proof. The proof of Theorem 3.6 in [6], while containing all of the essential strategy for proving satisfaction in the case Q_2, abbreviates the proof in two major ways. First, it only focuses on showing one direction of the containment in ($Q_{2n}(A)$: if). Second, and more significantly, the original proof outlined the basic strategy to take for building D in response to a given B and guess at a split S_i. However, since the set D constructed must work for every split S_i, it is not obvious how to avoid conflict between the basic strategies. Here we provide an explicit approach to addressing this issue.

Let A and C be the sets constructed in Lemma 4.2. To show that A satisfies $Q_{2n}(A)$ with $C \sqsupseteq_m A$, we fix an arbitrary $\vec{B} = (B_1, \ldots, B_n)$ as in $Q_{2n}(A)$ and construct $\vec{D} = (D_1, \ldots, D_n)$ in response. Furthermore, for every $S_j \sqsubset C$, we must describe $\vec{T^j} = (T^j_1, \ldots, T^j_n)$ in response. We fix an effective enumeration $\{(S_j, \hat{S}_j) \mid j \in \omega\}$ containing all disjoint pairs of c.e. subsets of C requiring, by way of the Slowdown Lemma ([9] p. 284), that the indices we list satisfy $S_j \cup \hat{S}_j = C \searrow (S_j \cup \hat{S}_j)$ in the canonical stagewise enumeration of c.e. sets.

We first give some intuition about the construction. We construct the sets $\vec{D}$ and $\vec{T^j}$ using the recursion theorem operating on for an index for $\vec{D}$. Given $\vec{D}$, $\vec{T^j}$, and S_j, suppose $S_j \sqsubset C$ and ($Q_{2n}(A)$: if). We will define a nondecreasing computable function p_j and consider the corresponding $X^{2n}_e = \overline{A}$ witnessing the $2n$-tardiness of A. If $x \in S_j \cap D_i$, then ($Q_{2n}(A)$: if) guarantees that x will enter A or B_i. Hence, the function p_j that measures the time it takes an element in $D_i \cap S_j$ to enter either A or B_i is computable. We essentially define T^j_i to be X^{2n}_{2i-1} and D_i to be X^{2n}_{2i} for $1 \leq i \leq n$. Thus, $\vec{T}$ can be thought of as elements that may stay out of A and $\vec{D}$ as elements that may enter A. The tardiness property of $X^{2n}_e = \overline{A}$ with respect to p_j then ensures that any x in $A \cap S_j \cap T_i$ must first enter either B_i or A before entering A. Hence, x enters B_i, and we use this fact to show that ($Q_{2n}(A)$: then) holds.

Since $\vec{D}$ is not built in response to the choice of $S_j \cup \hat{S}_j$, we split up the construction of D_i so that, on $T^j_1 \cap S_j$, the construction responds to S_j. The sets S_j are not disjoint but the approach remains valid as they make compatible demands. Our strategy only works when $(\vec{D}, \vec{B}, S_j)$ really satisfy ($Q_{2n}(A)$: if) and even then we must locate the correct $2n$-c.e. set X^{2n}_e with respect to p_j. We manage this complexity using a Π^0_2 guessing procedure at the true path $f(j)$ described below. Recall that A and C are fixed from Lemma 4.2, and $\vec{B}$ is fixed and arbitrary. We will formally define p_j later. We fix our tree to be $(w \cup \{\infty\})^{<\infty}$ and use $<_L$ to denote the lexicographic order as usual. We define the true path f by

$$f(j) = \begin{cases} \infty & \text{if } (S_j \sqcup \hat{S}_j) \neq C \vee \text{ ($Q_{2n}(A)$: if) fails for } (\vec{D}, \vec{B}, S_j) \\ e & \text{else for } e \text{ least s.t. } X^{2n}_e \text{ satisfies Lemma 4.2 w.r.t. } p_{f\restriction j}. \end{cases}$$

Given $\alpha \in (w \cup \{\infty\})^{<\infty}$, we let $X^\alpha = X^{2n}_{\alpha(|\alpha|)}$, $S_\alpha = S_{|\alpha|}$ and $\vec{T^\alpha}$ be the sets $\vec{T}$ built in response to S_α and X^α at α. Note that the property $(S_j \sqcup \hat{S}_j) = C$, the property ($Q_{2n}(A)$: if) holds for $(\vec{D}, \vec{B}, S_j)$, and the property X^{2n}_e satisfies Lemma 4.2 with respect to $p_{f\restriction j}$ are all Π^0_2 conditions. Hence, $f = \liminf_s f_s$ for a computable sequence $\{f_s \mid f_s \in (w \cup \{\infty\})^{<\infty}\}$, and $\alpha \prec f$ iff α is the $<_L$ string of length $|\alpha|$ satisfying $(\exists^\infty s)\ [f_s \succeq \alpha]$. Moreover, we assume that, if $f_s \succeq \alpha$, then it appears that $(S_j \sqcup \hat{S}_j) = C$, ($Q_{2n}(A)$: if) holds for $(\vec{D}, \vec{B}, S_j)$, and X_α satisfies Lemma 4.2 w.r.t. $p_{f\restriction j}$. For every x, let $\Gamma(x, s)$ denote the leftmost substring of

f_t of length x for $t \in [x, s]$. Note that if $\alpha \preceq f$, then $\Gamma(x, \infty) \succeq \alpha$ for all but finitely many x. Before we define p_α, we describe $\vec{D}$ and $\vec{T^\alpha}$. We need a few other notions to make these definitions.

$$x \in T^\alpha_{1,s+1} \leftrightarrow x \in T^\alpha_{1,s} \vee \left[s + 1 \geq x \wedge f_s \succeq \alpha \wedge x \in X^\alpha_{1,s} - C_s\right] \tag{4.7a}$$

$$s_x = (\mu t)(x \in C_t) \tag{4.7b}$$

$$\alpha_x = (\mu \beta \preceq \Gamma(x, s_x))(\exists t) \left[[x \in T^\beta_{1,s_x} \cap S_{\beta,t}] \wedge (\forall \gamma \prec \beta)[x \notin T^\gamma_{1,s_x} \vee x \in \hat{S}_{\gamma,t}]\right] \tag{4.7c}$$

$$x \in T^\alpha_{i+1} \leftrightarrow (\exists s) \left(f_s \succ \alpha_x \wedge x \in X^{\alpha_x}_{2i+1,s} \cap T^\alpha_{1,s}\right) \tag{4.7d}$$

$$x \in D_i \leftrightarrow (\exists s) \left(f_s \succ \alpha_x \wedge x \in X^{\alpha_x}_{2i,s}\right). \tag{4.7e}$$

The stage s_x is the least stage at which x enters C. The node $\alpha_x \preceq \Gamma(x, s_x)$ is the node (if it exists) so that x is in both $T^{\alpha_x}_1$ and S_{α_x}, and no shorter node can have this property.

Lemma 4.8. *If $\alpha \prec f$ and $f(|\alpha|) \neq \infty$, then $T^\alpha_1 \supseteq \overline{C}$, $T^\alpha_{i+1} \subseteq T^\alpha_i$ and $D_{i+1} \subseteq D_i$.*

Proof. The second and third claims follow from the construction and nesting of the sets $X^{\alpha_x}_i$ at stages where $\alpha \prec f_s$. The first claim follows from the fact that $X^\alpha_1 \supseteq \overline{A} \supseteq \overline{C}$ and $\alpha \prec f_s$ for infinitely many s. $\square$

Note that given $x \in C_s$ for $\alpha \prec f$, it is computable to determine whether $\alpha_x \preceq \alpha$ since $S_j \cup \hat{S}_j$ is actually a split of C for $\beta \preceq \alpha$. With this in mind, we define $p_\alpha(s)$. For every x and s, let

$$p_\alpha(i, x, s) = \begin{cases} s & \text{if } x \notin C_s \vee x \notin X^{\alpha_x}_{2i,s+1} \vee \alpha_x \npreceq \alpha \\ (\mu s' \geq s)(\alpha \preceq f_s \wedge x \in B_{i,s'} \cup A_{s'}) & \text{otherwise.} \end{cases}$$

Then let

$$p_\alpha(s) = 1 + \max_{\substack{x \leq s \\ i \leq n}} p_\alpha(i, x, s).$$

Lemma 4.9. *If $\alpha \prec f$, then $p_\alpha(s)$ is a total function.*

Proof. It suffices to show that $p_\alpha(i, x, s)$ is defined for all $x \leq s$ and $i \leq n$. If $x \notin C_s$ this is clear. If $x \in C_s$, we can computably check whether $\alpha_x \npreceq \alpha$ and hence whether $x \notin X^{\alpha_x}_{2i,s+1}$. Thus, we may assume that $p_\alpha(i, x, s)$ is defined by way of its second clause and $\alpha_x \preceq \alpha$. By definition of D_i and α_x, if $x \in X^{\alpha_x}_{2i,s+1}$ then $x \in D_i \cap S_{\alpha_x}$. Since $\alpha_x \preceq \alpha \prec f$, ($Q_{2n}(A)$: if) holds for $(\vec{D}, \vec{B}, S_{\alpha_x})$ so either $x \in A$ or $x \in B_i$. Hence, $p_\alpha(i, x, s)$ is defined by its second clause, and $p_\alpha(s)$ is a total function. $\square$

Lemma 4.10. *If $\alpha \prec f$, then for all but finitely many x,*

$$x \in T^\alpha_1 \cap S_\alpha \rightarrow \alpha_x \preceq \alpha.$$

Proof. Let t be a stage such that $\alpha \leq_L f_{t'}$ for all $t' \geq t$. Let $x > t$, and let s be the first stage at which $x \in T^\alpha_{1,s}$. Then, $s_x > s \geq x \geq t$ since $x \notin C_s$ by definition of $T^\alpha_{1,s}$. For $x > t$, we have $\alpha \leq_L \Gamma(x, s_x)$ and $\alpha \leq_L \alpha_x$ as well. Since $f_s \succeq \alpha$, we have that $\Gamma(x, s) \succeq \alpha$. Since $\Gamma(x, s_x) \leq_L \Gamma(x, s)$, we can conclude that $\alpha_x \preceq \Gamma(x, s_x)$ must either extend α or be a substring of α. Since $x \in T^\alpha_1 \cap S_\alpha$, we have that $\alpha_x \not\succ \alpha$ by the second conjunct in the definition of α_x. Thus, $\alpha_x \preceq \alpha$ for all sufficiently large $x \in T^\alpha_1 \cap S_\alpha$. $\square$

Lemma 4.11. *If $\alpha \prec f$, then $A \cap S_\alpha \cap T^\alpha_i \subseteq^* B_i \cap S_\alpha \cap T^\alpha_i$*

Proof. By Lemma 4.10, $\alpha_x \subseteq \alpha$ for all but finitely many $x \in T^\alpha_1 \cap S_\alpha$. Take such an x in $A \cap S_\alpha \cap T^\alpha_i$. By definition, $x \in S_{\alpha_x} \cap T^{\alpha_x}_1$. Since $x \in T^\alpha_i$, it follows that $x \in X^{\alpha_x}_{2i-1}$. Since $x \in A$ and $\alpha_x \prec f$, we also have $x \in X^{\alpha_x}_{2i}$. By (4.3e), if $s + 1$ is the least stage such that $x \in X^{\alpha_x}_{2i,s+1}$, then $x \in X^{\alpha_x}_{2i-1,s}$. Furthermore, by (4.3c), $x \in C_s$ so $p_{\alpha_x}(i, x, s)$ is defined by way of the second clause.

By Lemma 4.9, we know that $t = p_{\alpha_x}(i, x, s)$ is well defined hence either $x \in A_t$ or $x \in B_t$. However, $p_{\alpha_x}(s) \geq p_{\alpha_x}(i, x, s)$ and $x \in X^{\alpha_x}_{2i-1,s}$ so $x \notin A_t$. Hence $x \in B$. $\square$

Lemma 4.12. *If $\alpha \prec f$, then*

$$\begin{bmatrix} (A \cup T^\alpha_2) \cap (S_\alpha \cap T^\alpha_1) & \subseteq^* & B_1 \cap (S_\alpha \cap T^\alpha_1) \\ (A \cup T^\alpha_3) \cap (S_\alpha \cap T^\alpha_2) & \subseteq^* & B_2 \cap (S_\alpha \cap T^\alpha_2) \\ & \cdots & \\ (A \cup T^\alpha_n) \cap (S_\alpha \cap T^\alpha_{n-1}) & \subseteq^* & B_{n-1} \cap (S_\alpha \cap T^\alpha_{n-1}) \\ A \cap (S_\alpha \cap T^\alpha_n) & \subseteq^* & B_n \cap (S_\alpha \cap T^\alpha_n) \end{bmatrix}.$$

Proof. By Lemma 4.11, the last clause is established. Hence, let x be in $(A \cup T^\alpha_{i+1}) \cap (S_\alpha \cap T^\alpha_i)$ for $i < n$ so that $\alpha_x \preceq \alpha$ by Lemma 4.10. Again by Lemma 4.11, the result holds for all such x except for $x \in T^\alpha_{i+1} - A$. So, take such an $x \in T^\alpha_{i+1} - A$. Since $x \in T^\alpha_{i+1}$, it follows that $x \in X^{\alpha_x}_{2i+1}$. By (4.3d), we have $x \in X^{\alpha_x}_{2i}$ so $x \in D_i$. Since $\alpha \prec f$, we have $B_i \cap (S_\alpha - A) = D_i \cap (S_\alpha - A)$ and $x \in D_i \cap (S_\alpha - A)$. Hence, $x \in B_i$, completing the proof. $\square$

Lemma 4.13. *If $\alpha \prec f$, then*

$$\begin{bmatrix} (A \cup T_2^\alpha) \cap (S_\alpha \cap T_1^\alpha) & \supseteq^* & B_1 \cap (S_\alpha \cap T_1^\alpha) \\ (A \cup T_3^\alpha) \cap (S_\alpha \cap T_2^\alpha) & \supseteq^* & B_2 \cap (S_\alpha \cap T_2^\alpha) \\ & \dots & \\ (A \cup T_n^\alpha) \cap (S_\alpha \cap T_{n-1}^\alpha) & \supseteq^* & B_{n-1} \cap (S_\alpha \cap T_{n-1}^\alpha) \\ A \cap (S_\alpha \cap T_n^\alpha) & \supseteq^* & B_n \cap (S_\alpha \cap T_n^\alpha) \end{bmatrix}.$$

Proof. Assume $x \in B_i \cap (S_\alpha \cap T_i^\alpha)$, and by Lemma 4.10 suppose that $\alpha_x \preceq \alpha$. If $x \in A$, we are done, so suppose not. Since T_i^α is non-empty and ($Q_{2n}(A)$: if) is satisfied, then we have $x \in D_i$ as $x \in B_i \cap (S_\alpha - A)$. Thus, by (4.7e) we have $x \in X_{2i}^{\alpha_x}$. If $i = n$ then $x \in X_{2n}^{\alpha_x} \subset A$ since $\alpha_x \preceq \alpha \prec f$, contradicting our original assumption. For $i < n$, if $x \notin A$ we must have $x \in X_{2i+1}^{\alpha_x}$ by nesting. Since $x \in T_i^\alpha \subset T_1^\alpha$, (4.7d) entails that $x \in T_{i+1}^\alpha$. Hence, $x \in (A \cup T_{i+1}^\alpha) \cap (S_\alpha \cap T_i^\alpha)$, completing the lemma. □

We now finish demonstrating that A satisfies $Q_{2n}(A)$. Given any $\vec{B}$, we respond by building $\vec{D}$ as above. By Lemma 4.8, $D_{i+1} \subseteq D_i$. Given $S_j \sqsubset C$, we are done if $(\vec{B}, \vec{D}, S_j)$ does not satisfy ($Q_{2n}(A)$: if). Otherwise, let $\alpha \prec f$ have length j. By Lemma 4.8, we have $T_1^\alpha \supseteq \overline{C}$ and $T_{i+1}^\alpha \subseteq T_i^\alpha$. Lemmas 4.12 and 4.13 guarantee that $(S_j, \vec{T^\alpha}, \vec{B})$ satisfy ($Q_{2n}(A)$: then) modulo finite sets. To ensure that ($Q_{2n}(A)$: then) holds exactly, remove the elements in $S_\alpha \cap T_1^\alpha$ that violate the equalities in ($Q_{2n}(A)$: then) or the subset properties from T_i^α for all i. Only $T_1 \supseteq \overline{C}$ risks being violated by these modifications. All the elements removed from T_i^α, however, are elements of $S_\alpha \subseteq C$, so $T_1^\alpha \supseteq \overline{C}$ after these modifications.

Thus, for arbitrary $\vec{B}$, there is some $\vec{D}$ so that for all $S_j \sqsubset C$ either ($Q_{2n}(A)$: if) is unsatisfied or there is some properly nested $\vec{T}$ witnessing the satisfaction of ($Q_{2n}(A)$: then). Hence, $Q_{2n}(A)$ is satisfied by a properly $2n$-tardy. This completes the proof of Theorem 4.1, and we may conclude that there are infinitely many incomplete orbits. □

5. A low$_2$ and simple very tardy

Previously Harrington and Soare established the following theorem in [5].

Theorem 5.1 (*Harrington and Soare, Theorem 11.10 [5]*)**.** *If A is low (or even semilow) and simple then A is almost prompt.*

Since a very tardy set is simply one that is not almost prompt, this theorem shows that no very tardy can be both low and simple. Harrington and Soare's proof demonstrates that if A is very tardy and semilow there is a computable function that grows fast enough so that the corresponding n-c.e. complement of A is forced to leave an infinite c.e. set in $\overline{A}$. We show that Theorem 5.1 cannot be improved by constructing an example of a low$_2$ simple very tardy set. This example provides a negative answer to Question 1.7 of Harrington and Soare. We first offer a sketch of the tension in Theorem 5.1 so as to motivate the construction of our example.

Building a low set requires that we eventually preserve computations of the form $\Phi_{e,s}^{A_s}(e)$, while simplicity requires that if $W_{i,s}$ continues to grow, we eventually enumerate one of its members into A. Normally, we build a low simple set by only allowing a finite number of computations $\Phi_{e,s}^{A_s}(e)$ to restrain elements we see enter W_i out of A. However, building a very tardy set requires that we announce our intention to enumerate some element y into A long in advance. During the intervening time, a computation $\Phi_{e',s}^{A_s}(e')$ might converge and impose a restraint that y is obligated to respect. Hence, y must abandon its previously announced intention to enter A. If A was meant to be 2-tardy, this alone would cause a failure since 2-tardy sets cannot revoke their announced intentions to place elements into A. It might seem, on the other hand, that if we only aim to build a very tardy we could simply choose to leave y out of A and wait for another chance to place an element from W_i into A. However, by the time we observe that some y_n enters W_i, some later W_{i_n} may have already attempted to enumerate y_n into A and abandoned that attempt in response to a restraint from some computation $\Phi_{e_n,s}^{A_s}(e_n)$. Indeed, each y_n entering W_i may have already exhausted its guesses about entering A so that W_i no longer has the opportunity to place y into A.

It is clear from the above discussion that the need to restrain elements from entering A creates the potential to 'use up' the elements of some infinite c.e. set before we have the chance to place one of its members into A. Since lowness requirements in general require imposing some kind of restraint, it is interesting to see that Harrington and Soare's result fails for a weaker notion of lowness.

Theorem 5.2. *There is a simple 2-tardy set A that is low$_2$, i.e., satisfies $A'' \leq_T \emptyset''$.*

We construct A using a pinball machine laid out on top of a tree, in a manner similar to the one used to prove Theorem 2.1, however here our tree will be $2^{<\omega}$. Our construction will satisfy a version of the following three requirements.

$$\text{If } \varphi_e \text{ total \& nondecreasing, then } \big(\exists X_e^2\big)\big(X_e^2 = \overline{A} \wedge (\forall x)(\forall s)\big[x \in X_{e,s}^2 \implies x \notin A_{\varphi_e(s)}\big]\big) \tag{$\mathcal{N}_e$:}$$

$$|W_e| = \infty \implies A \cap W_e \neq \emptyset \tag{$\mathcal{P}_e$:}$$

$$\text{If } \limsup_s |W_{e,s}^{A_s}| = \infty \implies |W_e^A| = \infty \tag{$\mathcal{R}_e$:}$$

As before, requirement $\mathcal{N}_e$ ensures that A is 2-tardy. Requirement $\mathcal{P}_e$ guarantees that A is simple, and requirement $\mathcal{R}_e$ ensures that $A'' \leq_T \emptyset''$ by the following observation. Notice that

$$\limsup_s |W_{e,s}^{A_s}| = \infty \iff |\{\langle s, n\rangle \mid s = \mu t(|W_{e,t}^{A_t}| \geq n)\}| = \infty.$$

This fact and the fact that $\left\{e \middle| |W_e^A| = \infty\right\} \equiv_T A''$ guarantee that $A'' \leq_T \emptyset''$. In order to make our tree construction work, we will need to make a minor modification of $\mathcal{R}_e$. In the tree construction, we will use node γ to meet $\mathcal{N}_e$, node α to meet $\mathcal{P}_e$, and node β to meet $\mathcal{R}_e$, and we assign these requirements to nodes on the tree in the usual manner.

5.1. Tree argument

As in Theorem 2.1, we think of our tree $2^{<\omega}$ as growing upward with elements (balls) flowing down the tree towards the set A. As before, we define an approximation f_s to the true path f, where $f = \liminf_s f_s$, and balls will be placed on the pinball machine by a node $\alpha \prec f_s$ at stage s only when we wish to put them into A. At stage s, if ball x is on the machine, we denote the node at which it is located by $\alpha(x, s)$. If $\alpha(x, s) = \lambda$ (λ is the empty node), we put x into A_{s+1} and remove x from the machine at stage $s + 1$. We will sometimes decide later to remove balls from the machine so that they may not enter A. If $f_{s+1} <_L \alpha(x, s)$ we will also remove x from the machine at stage $s + 1$. At stage $s + 1$, we are free to place any ball $x \leq s + 1$ on the machine, as long as the following condition holds. For all s, if $\alpha(x, s)\downarrow$ then, for all t such that $x \leq t \leq s$, we have $\alpha(x, s) \leq_L f_t$, i.e., node $\alpha(x, s)$ is not reset between stage x and stage s. Note that this implies that there is no $t < s$ with $\alpha(x, t) <_L \alpha(x, s)$. This ensures that nodes cannot recycle balls that have been to their left, or equivalently, a node can recycle a ball only when that node has never been in a position to notice that the ball was used previously. We adopt the convention that the use of $W_{\alpha,s}$ is no more than s. Hence, the action of the machine ensures that if $\alpha \prec f_s$ then no α' with $\alpha <_L \alpha'$ can disrupt this computation because α' was reset and any new balls that might be placed in A for α' will be greater than s.

5.1.1. The $\mathcal{N}_\gamma$ module

We satisfy $\mathcal{N}_\gamma$ in the same way as in Theorem 2.1. Assuming φ_γ is computable and nondecreasing, requirement $\mathcal{N}_\gamma$ constructs a 2-c.e. set X_γ^2 that witnesses that A is 2-tardy with respect to φ_γ. The requirement $\mathcal{N}_\gamma$ builds the set X_γ^2 as follows. At stages s where $\gamma \preceq f_s$, we enumerate every $x < s$ with $\alpha(x, s) \npreceq \gamma$ and $x \notin A_s$ into $X_{\gamma_1}^2$. Also, whenever an element x is placed at γ, it is enumerated into $X_{\gamma_2}^2$. If $f_s <_L \gamma$, we restart the construction of X_γ^2, setting $X_{\gamma_1}^2 = X_{\gamma_2}^2 = \emptyset$.

Let $l_\gamma(s) = \max x[(\forall z < x)\varphi_{\gamma,s}(z)\downarrow]$. Given a stage $s + 1$ such that $\gamma \preceq f_{s+1}$, let $t \leq s$ be the greatest stage such that $\gamma \preceq f_t$ (if t does not exist let $t = 0$). If $l_\gamma(s) > l_\gamma(t)$ and, for all $x \in X_{\gamma_2,s}^2$, if $x \in X_{\gamma_2,\text{at } s'}^2$ then $l_\gamma(s) > s'$, then we say that $s + 1$ is γ-*expansionary*. At γ-expansionary stages $s + 1$, we move all balls x such that $\alpha(x, s) = \gamma$ downward so that $\alpha(x, s + 1) = \beta$, where $\beta\hat{\ }0 \preceq \gamma$ and β is the greatest such subnode of γ assigned to some $\mathcal{N}_{e'}$ or if no such β exists let $\beta = \lambda$. If $s + 1$ is γ-expansionary, we let $\gamma\hat{\ }0 \prec f_{s+1}$. Otherwise, we let $\gamma\hat{\ }1 \prec f_{s+1}$. The net effect of the definition of f_s is that if $t < s$ is the last stage where some ball was placed at γ then $\gamma\hat{\ }0 \npreceq f_s$ unless $\varphi_{\gamma,s}(t)\downarrow$. Thus, the ball is not released downward until the delay demanded by $\mathcal{N}_\gamma$ after entering $X_{\gamma_2}^2$ has expired. If we have moved any balls downwards or $|\gamma| = s$, we end this stage. Otherwise, we consider the action of $\gamma\hat{\ }0$ or $\gamma\hat{\ }1$. If $\gamma \prec f$, it is straightforward to show that $f \upharpoonright (|\gamma| + 1) = \liminf f_s \upharpoonright (|\gamma| + 1) = \gamma\hat{\ }0$ iff φ_γ is total.

5.1.2. The $\mathcal{P}_\alpha$ module

The action of $\mathcal{P}_\alpha$ tries to place some element from W_α into A. However, the elements $\mathcal{P}_\alpha$ directs towards A must respect $\mathcal{R}_\beta$ and $\mathcal{N}_\gamma$ for $\beta, \gamma \preceq \alpha$. Each $\mathcal{R}_\beta$ will have its own restraint function $r_\beta(\alpha, s)$ given below, and we define

$$r(\alpha, s) = \max_{\beta \prec \alpha} r_\beta(\alpha, s).$$

Suppose $\alpha \preceq f_{s+1}$. Let $\alpha\hat{\ }0 \preceq f_{s+1}$ if $A_s \cap W_{\alpha,s} \neq \emptyset$, and let $\alpha\hat{\ }1 \preceq f_{s+1}$ otherwise. If $A_s \cap W_{\alpha,s} = \emptyset$ and there is an $x > r(\alpha, s + 1)$ so that $x \in W_{\alpha,s+1}$, requirement $\mathcal{P}_\alpha$ places x on the largest node $\gamma \prec \alpha$ that corresponds to some tardiness requirement $\mathcal{N}_\gamma$ with $\gamma\hat{\ }0 \prec \alpha$. In this case or if $|\alpha| = s$, we end the stage. Otherwise, we consider the action of $\alpha\hat{\ }0$ or $\alpha\hat{\ }1$. If $\alpha \prec f$, it is clear that $f \upharpoonright (|\gamma| + 1) = \liminf f_s \upharpoonright (|\alpha| + 1) = \alpha\hat{\ }0$ if $A \cap W_\alpha \neq \emptyset$. Using properties of the restraint functions for $\mathcal{R}_\beta$, we will see that W_α is finite iff $f \upharpoonright (|\gamma| + 1) = \liminf f_s \upharpoonright (|\alpha| + 1) = \alpha\hat{\ }1$.

5.1.3. The $\mathcal{R}_\beta$ module

If $\limsup_s |W_{\beta,s}^{A_s}| = \infty$, requirement $\mathcal{R}_\beta$ must act to ensure that $|W_\beta^A| = \infty$. As previously discussed, satisfying this requirement for all β guarantees that A is low$_2$. The role of $\mathcal{R}_\beta$ is to define a restraint function that preserves computations of the form $|W_{\beta,s}^{A_s}| = n$. Let $l_\beta(s) = |W_{\beta,s}^{A_s}|$. Given a stage $s + 1$ such that $\beta \preceq f_{s+1}$, let $t \leq s$ be the greatest stage such that $\beta \preceq f_t$ (if t does not exist, let $t = 0$). If $l_\beta(s + 1) > l_\beta(t)$, then we say that $s + 1$ is β-*expansionary*. If $s + 1$ is β-expansionary, we let $\beta\hat{\ }0 \prec f_{s+1}$. Otherwise, we let $\beta\hat{\ }1 \prec f_{s+1}$.

If $\beta \prec f_s$, we wish to preserve the computations that put elements into $W^{A_s}_{\beta,s}$, so as to preserve the cardinality of this set. The following timing functions will help us describe the restraint sufficient to satisfy $\mathcal{R}_\beta$. For $n \leq l_\beta(s)$, let

$$u^\beta_s(n) = \mu t \leq s\,(\exists x_1 \neq x_2 \neq \ldots \neq x_n)\,(\forall j \leq n)\big(\forall t' \in [t, s]\big)[x_j \in W^{A_{t'}}_{\beta,t'}].$$

If $\beta \preceq f_s$ set

$$r_\beta(\delta, s) = \begin{cases} 0 & \text{if } \delta \not\succeq \beta\hat{\ }0\text{, otherwise} \\ s & \text{if } |\delta| - |\beta| > l_\beta(s) \\ u^\beta_s(|\delta| - |\beta|) & \text{else.} \end{cases}$$

Notice that no restraint is placed on nodes $\delta \not\succeq \beta\hat{\ }0$. Our convention about the use of $|W^{A_s}_{\beta,s}|$ and the action of the tree guarantees that no $\mathcal{P}_\delta$ for $\delta \not\succeq \beta\hat{\ }0$ is able to remove elements from $W^{A_s}_{\beta,s}$. On the other hand, the restraint function ensures that there are only finitely many nodes that might disrupt the computations responsible for enumerating the n longest residing elements of $W^{A_s}_{\beta,s}$ for $n \leq l_\beta(s)$. We will demonstrate that this restraint ensures the following variant of $\mathcal{R}_\beta$ is satisfied for $\beta \prec f$.

$$\limsup_{s\,|\,\beta \preceq f_s} |W^{A_s}_{\beta,s}| = \infty \implies |W^A_\beta| = \infty. \qquad (\mathcal{R}_\beta)$$

The requirement $\mathcal{R}_\alpha$ differs in that $\limsup_s |W^{A_s}_{\beta,s}|$ is only evaluated at stages s such that $\beta \preceq f_s$. This avoids any transitory effects that might enumerate elements into $W^{A_s}_{\beta,s}$ and then remove them again before $\mathcal{R}_\beta$ has a chance to act. As we will see, however, it is no less effective a means to show that $A'' \equiv_T \emptyset''$. We now show that this construction ensures that all the requirements are satisfied.

5.2. Verification

We observe three important properties of our construction.

Lemma 5.3. 1. *If $\alpha \preceq f_s$, there are no balls at any $\beta \prec \alpha$ not in A.*
2. *If $f_s \succeq \alpha(x, s-1)\hat{\ }0$ then $\alpha(x, s) \prec \alpha(x, s-1)$, provided $\alpha(x, s-1)$ and $\alpha(x, s)$ are defined.*
3. *If $\alpha \prec f$, then there is a stage s such that for all $t \geq s$ we have $\alpha \prec f_t$.*

The first two follow directly from the action of the tree, and the third property follows immediately from the definition of f. We now show that the true path is total. We first note that no individual ball ever stalls on the true path.

Lemma 5.4. *If $\alpha(x, s) = \gamma$ and $f_t \succeq \gamma\hat{\ }0$ where $t > s$, then either the node γ was reset between stage t and s or $x \in A_t$.*

Proof. Since balls are placed only on nodes for negative requirements, node γ works for $\mathcal{N}_\gamma$. By Lemma 5.3 Statement 2 and the tree construction, the only way x can leave γ is if γ is reset or x moves to a predecessor of γ. If γ is not reset between stage s and t, ball x must reach A by induction and the $\mathcal{N}_\gamma$ module. □

We need to know that no positive requirement emits so many balls that f_s cannot extend a particular node.

Lemma 5.5. *If $\alpha \preceq f$ then $\mathcal{P}_\alpha$ places at most finitely many balls on the tree. Furthermore, if $\alpha \preceq f$ and $\beta \succ \alpha$, then only finitely many balls placed on the tree by $\mathcal{P}_\beta$ travel down the tree to reach α.*

Proof. If α places infinitely many balls on the tree, then there is a stage s and a ball $x \in W_{\alpha,s}$ so that $\mathcal{P}_\alpha$ emits x at stage s and α is no longer reset after stage s. Furthermore, $\mathcal{P}_\alpha$ must emit another ball later, so there is a stage $t > s$ with $f_t \succeq \alpha$. Hence, $x \in A_t$ by 5.4. By the action of the $\mathcal{P}_\alpha$ module, however, $\mathcal{P}_\alpha$ ceases emitting once $x \in A_t \cap W_{\alpha,t} \neq \emptyset$. Thus, only finitely many balls are placed on the tree by $\mathcal{P}_\alpha$. The second half of the claim follows by the same argument applied to the stages at which x reaches α. □

Lemma 5.6. *$f = \limsup_s f_s$ is a path through $2^{<\omega}$.*

Proof. Suppose not; then $f = \delta$ for some $\delta \in 2^{<\omega}$ and $\{s \,|\, f_s = \delta\}$ is infinite. By construction, this occurs for large s only if every time $f_s = \delta$, there is some x with $\alpha(x, s) = \gamma \prec \delta$. Pick s_0 large enough so that δ is never reset after s_0, and no $\mathcal{P}_\beta$ with $\beta \preceq \delta$ places any ball on the tree after s_0. Now, pick some t and s_1 where $t > s_1 > s_0$ such that $f_t = f_{s_1} = \delta$ and no nodes extending δ are visited between stages t and s_1. If x is such that $\alpha(x, t) = \gamma \prec \delta$, the ball x cannot have come down from some node extending δ nor can it have been placed at γ after s_0. Hence, $\alpha(x, s_1) = \gamma$, violating Lemma 5.4, a contradiction. □

Before we can conclude that $\mathcal{P}_\alpha$ is satisfied, we first must argue that $\mathcal{R}_\beta$ imposes only finitary restraint on the true path f. We need two further lemmas. The first shows that the only way ball x can pass by ball y is if y is placed on the tree first and later x is added with $y <_L x$.

Lemma 5.7. *Suppose x is placed on the tree at stage s and y is not at a node extending $\alpha(x, s)$ at stage s. If x remains on the tree until stage $s' > s$ and $\alpha(y, s') \succeq \alpha(x, s')$, then $\alpha(y, s) <_L \alpha(x, s)$.*

Proof. If x remains on the tree at stage s', the construction guarantees that, for $s \le t \le s'$, nodes extending $\alpha(x, t)$ are not visited at stage t. So, y cannot be added to the tree above x at these stages t. Since balls move downward on the tree, if $t + 1 > s$ is the least stage at which $\alpha(y, t + 1) \succeq \alpha(x, t + 1)$, we have either $\alpha(x, t) <_L \alpha(y, t)$ or $\alpha(y, t) <_L \alpha(x, t)$. However, the motion of x at stage t requires that $f_{t+1} \succeq \alpha(x, t)$. So, if $\alpha(x, t) <_L \alpha(y, t)$, then y is removed from the tree at stage $t+1$, contradicting our assumption that $\alpha(y, t+1) \succeq \alpha(x, t+1)$. Therefore, $\alpha(y, t) <_L \alpha(x, t)$. Provided that the balls remain on the tree between s and t, we have $\alpha(y, t) \preceq \alpha(y, s)$ and $\alpha(x, s) \succeq \alpha(x, t)$. If y was added to the tree to the left of the location of x after x was placed on the tree, then x would have been removed from the tree. So, $\alpha(y, s) <_L \alpha(x, s)$. □

We now show the following. If $\beta \prec f$ then infinitely often the only balls above β are those that will never move below β without being reset. Let

$$B_t = \left\{x \,\middle|\, \alpha(x, t) \succ \beta \wedge \left(\exists t' > t\right)\left(\alpha(x, t') \preceq \beta\right)\right\}.$$

Let $\hat{B_t}$ be the set of elements in B_t that reach β without being reset between stages t and t' in the definition of B_t.

Lemma 5.8. *If $\beta \prec f$ then for every s there is a $t \ge s$ such that $f_t \succ \beta$ and $\hat{B_t} = \emptyset$.*

Proof. By Lemma 5.5 and the fact that $\beta \prec f$, we may assume without loss of generality that stage s is so large that β is never reset after s and that every positive requirement below β no longer acts. Suppose $\hat{B_s} \ne \emptyset$. Let $x \in \hat{B_s}$ be such that $\alpha(x, s)$ is minimal in $\hat{B}_s$ under $<_L$ and maximal among those elements under $\succ$. By definition of $\hat{B}_s$, there is a least stage $t > s$ at which $\alpha(x, t) \preceq \beta$. Now, given any $y \in \hat{B}_t$, we have $\alpha(y, t) \succ \beta \succ \alpha(x, t)$, and by maximality of $\alpha(x, s)$ under $\prec$, we know that y is not at a node extending $\alpha(x, s)$ at stage s. So, by Lemma 5.7, we know that $\alpha(y, s) <_L \alpha(x, s)$. Since neither x nor y was reset between stages s and t (by assumption about x and the movement of balls on the tree), the element y is an element in $\hat{B_s}$ to the left of x, a contradiction. Hence, $\hat{B_t} = \emptyset$. The second part of the lemma follows by choosing $t' \ge t$ to be the stage at which x enters A and again using Lemma 5.7 to show no balls from the right could get above the node at which x rests before it enters A. □

Recall $r(\alpha, s) = \max_{\beta \prec \alpha} r_\beta(\alpha, s)$ is the restraint imposed on $\mathcal{P}_\alpha$ by the $\mathcal{R}_\beta$ for $\beta \prec \alpha$.

Lemma 5.9. *If $\beta \prec f$, then $\mathcal{R}_\beta$ is satisfied, and if $\alpha \succ \beta$ is also on the true path f, then $\lim_{s\to\infty} r_\beta(\alpha, s)$ is finite. Hence, $\lim_{s\to\infty} r(\alpha, s)$ is finite as well.*

Proof. If $\limsup_s l_\beta(s) = |W^{A_s}_{\beta,s}| < \infty$, then $\mathcal{R}_\alpha$ is satisfied, and the second half follows trivially by definition of $r_\beta(\alpha, s)$ (since $\alpha \not\succeq \beta\hat{\ }0$). Otherwise, we work after a stage large enough so that β is not longer reset and no positive requirements below β place balls on the tree. We claim that for every n there is some stage s_n and elements $x_1, x_2, \dots, x_n$ satisfying

1. $f_{s_n} \succeq \beta$,
2. $(\forall\, i \ne j \le n)[x_i \ne x_j]$, and
3. $(\forall\, i \le n)(\forall\, t \ge s_n)[x_i \in W^{A_t}_{\beta,t}]$.

By the definition of $u^\beta_s(n)$, the last equation entails that $s_n \ge u^\beta_t(n)$ since element $x_n \in W^{A_t}_{\beta,t}$ for every $t \ge s_n$.

To verify the claim, suppose n is the least failure of this claim. By Lemma 5.5, we can pick $s > s_{n-1}$ large enough so that every ball placed on the machine by $\mathcal{P}_\alpha$ with $\alpha \succ \beta$ and $|\alpha| - |\beta| < n$ which will ever enter A has already done so. Now pick $s' > s$ with $f_{s'} \succ \beta$ and $\hat{B}_{s'} = \emptyset$ as given in Lemma 5.8. Finally, let $t \ge s'$ be the least stage with $f_t \succeq \beta\hat{\ }0$. Note that $\hat{B_t} = \emptyset$ as well. At stage t, the only balls above β are those added at this very stage and those that will be reset before they get below β.

By the strategy given for $\mathcal{R}_\beta$, we know that $l_\beta(t) > l_\beta(s) \ge n - 1$. Let x_n be the element that has occupied $W^{A_t}_{\beta,t}$ for the longest uninterrupted time and is not equal to any of x_m for $m < n$. In other words, x_n is the element (distinct from the x_m for $m < n$) in $W^{A_{t'}}_{\beta,t'}$ for all $t' \in [\hat{s}, t]$ for the least stage $\hat{s}$. Since $f_t <_L \beta\hat{\ }1$, no $\mathcal{P}_\alpha$ with $\alpha \succeq \beta\hat{\ }1$ can add balls less than t to A and, thus, cannot remove x_n from W^A_β. On the other hand, if $\alpha \succeq \beta\hat{\ }0$ and if $\mathcal{P}_\alpha$ places any elements in A after stage t, we must have $|\alpha| - |\beta| \ge n$ by our choice of t. This implies that, for all $t' \ge t$, if $x_n \in W^{A_{t'}}_{\beta,t'}$ and $\hat{s}$ is the first stage at which x_n entered W^A_β and remained in until t', then $r_\beta(\alpha, t') \ge \hat{s}$. Thus, the restraint guarantees any new balls placed on the tree cannot remove x_n from W^A_β. Moreover, any old balls already on the tree above α are reset before they get to α because $\hat{B_t} = \emptyset$, so they cannot remove x_n from W^A_β. Hence, choosing $s_n = t$ satisfies the claim, and it follows immediately that $\mathcal{R}_\beta$ is satisfied. We also have that $\lim_{s\to\infty} r_\beta(\alpha, s)$ is finite by definition of $r_\beta(\alpha, s)$ since $|\alpha| - |\beta| \le l_\beta(t')$ and $u^\beta_{t'}(|\alpha| - |\beta|) \le s_{|\alpha|-|\beta|}$ for every $t' \ge s_{|\alpha|-|\beta|}$. □

Lemma 5.10. *A is 2-tardy and simple.*

Proof. The sets built at the $\mathcal{N}_\gamma$ nodes along the true path f witness that A is 2-tardy. Suppose φ_γ is total and $\gamma \prec f$. Let s' be the last stage at which γ is reset. For each stage $s \geq s'$ such that $f_s \succ \gamma$, all elements $x < s$ with $x \notin A_s$ and $\alpha(x, s) \not\preceq \gamma$ are enumerated into $X^2_{\gamma_1}$. By construction, any ball in $X^2_{\gamma_1}$ that passes through node γ after stage s' enters $X^2_{\gamma_2}$ and is appropriately delayed by φ_γ before entering A.

Consider a ball $y \leq t$ on a node $\delta' \succeq \delta$ for $\delta \prec \gamma$ and $\gamma <_L \delta'$ at some stage $t > s'$ where $\delta' \prec f_t$ Let t' be the least stage greater than t such that $f_{t'} \succ \gamma$. Since δ' was reset at stage s', we have that $y \geq s'$ and hence y was not placed in $X^2_{\gamma_1}$ at stage s'. So, y does not violate the 2-tardy property if y already entered A via the node δ by stage t' as it will never enter X^2_γ by construction. Otherwise, y is reset by stage t' and is added to $X^2_{\gamma_1}$ at stage t'. At a later stage y may be placed on the tree, but only at a node extending γ since all nodes to the right of γ are reset at stage t'. So, if y later enters A it will be delayed the proper amount of time. Hence, we have that $X^2_\gamma =^* \overline{A}$, and we have met $\mathcal{N}_\gamma$.

By Lemma 5.9, if W_α is an infinite c.e. set, then eventually some element in W_α is greater than the finite value $\lim_{s\to\infty} r(\alpha, s)$ and enters W_α after the last stage at which α is reset. Thus, $\mathcal{P}_\alpha$ will succeed in making $A \cap W_\beta \neq \emptyset$. □

Lemma 5.11. $A'' \leq_T \emptyset''$.

Proof. Recall that requirement $\mathcal{R}_\beta$ guarantees that W^A_β is infinite if and only if $\limsup_s l_\beta(s) = \infty$. To determine whether W^A_β is infinite, $\emptyset''$ computes whether $\beta\hat{\ }0 \prec f$, i.e., whether there are infinitely many stages s such that s is a β-expansionary stage. Thus, $A'' \leq_T \emptyset''$. □

We have proved Theorem 5.2 by constructing a simple low$_2$ 2-tardy set.

6. Open questions

We know that any set that satisfies Q_n is n-tardy and not automorphic to a complete set. We do not know whether there are examples of n-tardy sets that are not automorphic to a complete set but also do not satisfy Q_n.

Question 6.1. *Does every n-tardy set not automorphic to a complete set satisfy Q_n?*

We would also like to know whether there is a properly very tardy set that is not automorphic to a complete set.

Question 6.2. *Is there a very tardy set that is not n-tardy for any n and is not automorphic to a complete set?*

The above question could be attacked using definable properties. We have not yet found a property that describes the properly very tardy sets, i.e., those very tardy sets that are not n-tardy for any $n \in \omega$.

Question 6.3. *Find a property Q_∞ so that if $Q_\infty(A)$ holds, then A is very tardy, and find some very tardy set A that is not n-tardy for any n and satisfies $Q_\infty(A)$.*

Finally, we want to know whether Theorem 2.1 can be extended as follows.

Question 6.4. *Is there a properly $n+1$-tardy set that is not computed by any n-tardy sets?*

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